Meaning of Noncommutative Geometry and the Planck-Scale Quantum Group

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This is an introduction for nonspecialists to the noncommutative geometric approach to Planck scale physics coming out of quantum groups. The canonical role of the 'Planck scale quantum group' $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft\mathbb{C}[p]$ and its observable-state T-duality-like properties are explained. The general meaning of noncommutativity of position space as potentially a new force in Nature is explained as equivalent under quantum group Fourier transform to curvature in momentum space. More general quantum groups $\mathbb{C}(G^\star)\blacktriangleright\!\!\triangleleft U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are also discussed. Finally, the generalisation from quantum groups to general quantum Riemannian geometry is outlined. The semi-classical limit of the latter is a theory with generalised non-symmetric metric $g_{\mu\nu}$ obeying $\nabla_\mu g_{\nu\rho}-\nabla_\nu g_{\mu\rho}=0$.

1 Introduction

There are currently several approaches to Planck-scale physics and of them 'Noncommutative geometry' is probably the most radical but also the least well-tested. As Lee Smolin in his lectures at the conference was kind enough to put it, it is 'promising but too early to tell'. Actually my point of view, which I will explain in these lectures, is that some kind of noncommutative geometry is *inevitable* whatever route we take to the Planck scale. Whether we evolve our understanding of string theory, compute quasiclassical states in loop-variable quantum gravity, or just investigate the intrinsic mathematical structure of geometry and quantum theory themselves (my own line), all roads will in my opinion lead to *some kind* of noncommutative geometry as the next more general geometry beyond nonEuclidean that is needed at the Planck scale where both quantum and gravitational effects are strong. I think the need for this and its general features can be demonstrated from simple nontechnical arguments and will try to do this here. These philosophical and conceptual issues are in Section 2.

Beyond this, and definitely a matter of opinion, it seems to me that there is are certain philosophical principles [1] which can serve as a guide to what Planck scale physics should be, in particular what I have called the *principle of representation-theoretic self-duality* (of which T-duality is one manifestation). I believe that to proceed one has to ask in fact what is the *nature of physical reality itself.* In fact I do not think that theoretical physicists can any longer

afford to shy away from such questions and, indeed, with proposals for Planck-scale physics beginning to emerge it is already clear that some new philosophical basis is going to be needed which will likely be every bit as radical as those that came with quantum mechanics and general relativity. My own radical philosophy in [2][3][1] basically takes the reciprocity ideas of Mach to a modern setting. But it also suggests a different concept of reality, which I call *relative realism*, from the reductionist one that most theoretical physicists are still unwilling to give up (I said it would be radical). This might seem fanciful but what it boils down to in practice is an extension of ideas of Fourier theory to the quantum domain. Section 3 provides a modern introduction to this.

Next I will try to convince you that while there are still several different ideas for what *exactly* noncommutative geometry should be, there is slowly emerging what I call the 'quantum groups approach to noncommutative geometry' which is already *fairly* complete in the sense that it has the same degree of 'flabbiness' as Riemannian geometry (is not tied to specific integrable systems etc.) while at the same time it includes the 'zoo' of already known naturally occurring examples, mostly linked to quantum groups. Picture yourself for a moment in the times of Gauss and Riemann; clearly spheres, tori, etc., were evidently examples of something, but of what? In searching for this Riemann was able to formulate the notion of Riemannian manifold as a way to capture known examples like spheres and tori but broad enough to formulate general equations for the intrinsic structure of space itself (or after Einstein, space-time). Theoretical physics today is in a parallel situation with many naturally occurring examples from a variety of sources and a clear need for a general theory. Our approach[4] is based on fiber bundles with quantum group fiber[5], and we will come to it by the end of the lectures, in Section 6. It includes a working definition of 'quantum manifold'.

In between, I will try to give you a sense of some noncommutative geometries out there from which our intuition has to be drawn. We will 'see the sights' in the land of noncommutative geometry at least from the quantum groups point of view. Just as Lie groups are the simplest Riemannian manifolds, quantum groups are the simplest noncommutative spaces. Their homogeneous spaces are also covered, as well as quantum planes (which are more properly braided groups). We refer to [6] for more on quantum groups themselves.

At the same time, the physics reader will no doubt also want to see testable predictions, detailed models etc. While, in my opinion, it is *still* too early to rush into building models and making predictions ('one cannot run before one can walk') I will focus on at least one toy model of Planck-scale physics using these techniques. This is the Planck-scale quantum group introduced 10 years ago in [3][2] and exhibiting even then many of the features one might consider important for Planck scale physics today, including duality. This is the topic of Section 4. It is not, however, the 'theory of everything' or M-theory etc. I seriously doubt that Einstein could have formulated general relativity without the mathematical definition of a 'manifold' having been sorted out by Riemann a century before (and which I would guess had filtered down to

Einstein's mathematical mentors such as Minkowski). In the same way, one really needs to sort out the correct or 'natural' definitions of noncommutative geometry some more (in particular the Ricci tensor and stress energy tensor are not yet understood) before making attempts at a full theory with testable predictions. This is on the one hand mathematics but on the other hand it has to be guided by physical intuition with or even without firm predictive models. In fact the structure of the mathematical possibilities of noncommutative geometry (which means for us results in the theory of algebra) can tell us a lot about any actual or effective theory even if it is not presently known.

The general family of *bicrossproduct quantum groups* arising in this way out of Planck scale physics contains many more examples (it is one of the two main constructions by which quantum groups originated in physics.) For example, there is a quantum group $\mathbb{C}(G^\star)\blacktriangleright\!\!\triangleleft U(\mathfrak{g})$ for every complex simple Lie algebra $\mathfrak{g}$. All these bicrossproduct quantum groups can be viewed as the actual quantum algebras of observables of actual quantum systems and can be viewed precisely as models unifying quantum and gravity-like effects [2][3]. For the record, the bicrossproduct construction $\blacktriangleright\!\!\triangleleft$ was introduced in this context at about the same time (in 1986) but independently of the more well-known quantum groups $U_q(\mathfrak{g})$ [7][8], in particular before I had even heard of V.G. Drinfeld or integrable systems. To this day the two classes of quantum groups, although constructed from the same data $\mathfrak{g}$, have never been directly related (this remains an interesting open problem). The situation is shown in Figure 1. To build a theory of noncommutative geometry we need to include naturally occurring examples such as these.

We also need to include the more traditional noncommutative algebras to which people have traditionally tried to develop geometric pictures, namely the canonical commutation relations algebra $[x,p]=\imath\hbar$ or its group version the Weyl algebra or 'noncommutative torus' $vu=e^{\imath\alpha}uv$ as in the work of A. Connes[9]. We can also consider the matrix algebras $M_n(\mathbb{C})$ as studied by Dubois-Violette, Madore and others; as we saw seen in the beautiful lecture of Richard Kerner at the conference, one can do a certain amount of noncommutative geometry for such algebras too. On the other hand, in some sense these are actually all the same example in one form or another, i.e. basically the algebra of operators on some Hilbert space (at least for generic α). These examples and the traditional ideas of vector fields as derivations and points as maximal ideals etc., come from algebraic geometry and predate quantum groups. In my opinion, however, one cannot build a valid noncommutative geometry always on the basis of essentially one example (and a lot of elegant mathematics) – one has to also include the rich vein of practical examples such as the quantum groups above. The latter have a much clearer geometric meaning but very few derivations or maximal ideals etc., i.e. we have to develop a much less obvious noncommutative differential geometry if we are to include them as well as the traditional matrix algebras and of course the commutative case corresponding to usual geometry. This is precisely what has emerged slowly in recent years and that which I will try to explain.

In Section 5 we turn for completeness, to the other and more well-known type of quantum

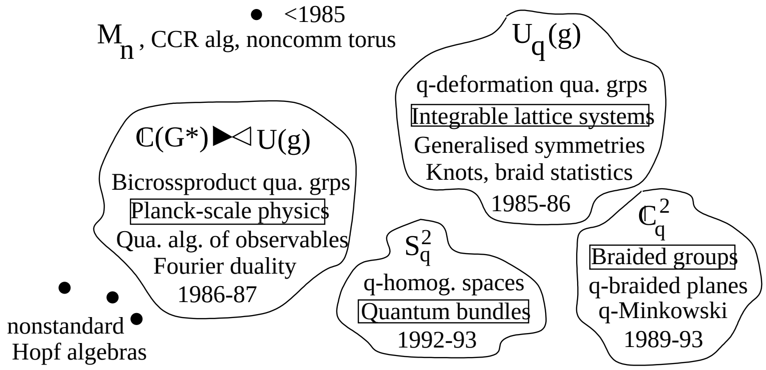

Figure 1: The landscape of noncommutative geometry today. Some isolated 'traditional' objects such as matrix algebras and the noncomm. torus, and two classes of quantum groups

groups, the q-deformed enveloping algebras $U_q(\mathfrak{g})$. These did not arise at all in connection with Planck scale physics or even directly as the quantisation of any physical system. Rather they are 'generalised symmetries' of quantum or lattice integrable systems. Nevertheless they are also examples of noncommutative geometry and, if recent conjectures of Lee Smolin and collaborators prove correct, they are natural descriptions of the noncommutative geometry coming out of the loop variable approach to quantum gravity. The more established meaning of *these* quantum groups which we will focus on is that they induce braid statistics on particles transforming as their representations. In effect the dichotomy of particles into bosons (force) and fermions (matter) is broken in noncommutative geometry and in fact both are unified with each other and with quantisation. Roughly speaking the meaning of q here is a generalisation of the -1 for supersymmetry. So the natural noncommutative geometry here is 'braided geometry'[10]. Yet at the same time one may write q in terms of Planck's constant or, according to [11], the cosmological constant. It means that one physical manifestation of quantum gravity effects is as braid (e.g. fractional) statistics.

Finally, more accessible perhaps to many readers will be not so much our proposals for the full noncommutative theory but its semiclassical predictions; in order to be naturally made noncommutative one has to shift ones point of view a little and indeed move to a slightly more general notion of classical Riemannian geometry. The main prediction is that one should replace the notion of metric and its Levi-cevita connection by a notion of nondegenerate 2-tensor (not necessarily symmetric) and the notion of vanishing torsion and vanishing *cotorsion*. The cotorsion tensor associated to a 2-tensor is a new concept recently introduced in [4]. The resulting self-dual generalisation of the usual metric compatibility becomes

$$\nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\mu\rho} = 0.$$

The generalisation allows a synthesis of symplectic and Riemannian geometry, which is a semi-classical analogue of the quantum-gravity unification problem. Not surprisingly, the above ideas

turn out to be related at the semiclassical level to other ideas for Planck scale physics such as T-duality for sigma models on Poisson-Lie groups[12], see [13].

2 The meaning of noncommutative geometry

It stands to reason that if one seriously wants to unify quantum theory and gravity into a single theory with a single elegant point of view, one must first formulate each in the same language. On the side of gravity this is perfectly well-known and we do not need to belabour it; instead of points in a manifold one should and does speak in terms of the algebra of its coordinate functions e.g. (locally) position coordinates $\mathbf{x}$, say. Geometrical operations can then be expressed in terms of this algebra, for example a vector field might be a derivation on the algebra. 'Points' might be maximal ideals. This conventional point of view (called *algebraic geometry*) doesn't really work in practice in the noncommutative case, i.e. it needs to be modified, but it is a suitable starting point for the unification.

What about quantum mechanics? Well this too is some kind of algebra, of course noncommutative due to noncommutation relations between position and momentum. So the language we need is that of algebras. We need to modify usual algebraic geometry in such a way that it extends to algebras of observables arising in quantum systems. At the same time we should, I believe, also be guided by finding natural mathematical definitions that both include nontrivial applications in mathematics *and* encode those algebras in quantum systems which have a clear geometrical structure self-evidently in need of being encoded (perhaps even without direct physical input about Planck scale physics). For example, before the discovery of quantum groups noncommutative geometry made only minimal changes in pursuit of the above idea, e.g. to let the algebra be noncommutative but nevertheless define a vector field as a derivation. All very elegant, but not sufficient to include 'real world' examples like quantum groups.

One other general point. For classical systems we frequently make use of deep classification and other theorems about smooth manifolds; the rich structure of what is mathematically allowed e.g. by topological constraints is often a guide to building effective theories even if we do not know the details of the underlying theory. If we accept the above then the corresponding statement is that deep mathematical theorems about the classification and structure of noncommutative algebras ought to tell us about the possible effective corrections from quantum gravity even before a full theory is known (as well as be a guide to the natural structure of the full theory). We will see this in some toy examples in the next chapter. By contrast many physicists seem to believe that the only algebra in physics is the CCR algebra (or its fermionic version), or possibly Lie algebras, as if there is not in fact a much richer world of noncommutative algebras for their theories to draw upon. In fact this noncommutative world has to be at least as rich as the theory of manifolds since it must contain them in the special commutative limit. I contend that the intrinsic properties of noncommutative algebras is where we should look for new principles and ideas for the Planck scale.

2.1 Curvature in momentum space – a possible new force of nature

Before going into details of the modern approach to noncommutative geometry we want to consider some general issues about unifying quantum theory and geometry using algebra. In particular, what finally emerges as the true meaning of noncommutative geometry for Planck scale physics? In a nutshell, the answer I believe is as follows. Thus, to survive to the Planck scale we should cling to only the very deepest ideas about the nature of physics. In my opinion among the deepest is 'Born reciprocity' or the arbitrariness under position and momentum. Now, in conventional flat space quantum mechanics we take the $\mathbf{x}$ commuting among themselves and their momentum $\mathbf{p}$ likewise commuting among themselves. The commutation relation

$$[x_i, p_j] = \imath\hbar\delta_{ij} \tag{1}$$

is likewise symmetric in the roles of $\mathbf{x}, \mathbf{p}$ (up to a sign). To this symmetry may be attributed such things as wave-particle duality. A wave has localised $\mathbf{p}$ and a particle has localised $\mathbf{x}$.

Now the meaning of *curvature* in position space is, roughly speaking, to make the natural conserved $\mathbf{p}$ coordinates noncommutative. For example, when the position space is a 3-sphere the natural momentum is su_2. The enveloping algebra $U(su_2)$ should be there in the quantum algebra of observables with relations

$$[p_i, p_j] = \frac{\imath}{R}\epsilon_{ijk}p_k \tag{2}$$

where R is proportional to the radius of curvature of the S^3.

By Born-reciprocity then there should be another possibility which is *curvature in momentum space*. It corresponds under Fourier theory to noncommutativity of position space. For example if the momentum space were a sphere with m proportional to the radius of curvature, the position space coordinates would correspondingly have noncommutation relations

$$[x_i, x_j] = \frac{\imath}{m}\epsilon_{ijk}x_k. \tag{3}$$

Mathematically speaking this is surely a symmetrical and equally interesting possibility which might have observable consequences and might be observed. Note that m here is just a parameter not necessarily mass, but our use of it here does suggest the possibility of understanding the *geometry of the mass-shell* as noncommutative geometry of the position space $\mathbf{x}$. This may indeed be an interesting and as yet unexplored application of these ideas. In general terms, however, the situation is clear: for systems constrained in position space one has the usual tools of differential geometry, curvature etc., of the constrained 'surface' in position space or tools for noncommutative algebras (such as Lie algebras) in momentum space. *For systems constrained in momentum space one needs conventional tools of geometry in momentum space or, by Fourier theory, suitable tools of noncommutative geometry in position space.*

In mathematical terms, these latter two examples (2),(3) demonstrate the point of view of noncommutative geometry: we are viewing the enveloping algebra $U(su_2)$ *as if* it were the

algebra of coordinates of some system, i.e. we want to answer the question

$$U(su_2) = C(?)$$

where ? will not be any usual kind of space (where the coordinates would commute). This is what we have called in [14] a 'quantum-geometry transformation' since a quantum symmetry point of view (such as the angular momentum in a quantum system) is viewed 'up-side-down' as a geometrical one. The simplest example $U(\mathfrak{b}_+)$ was studied from this point of view as a noncommutative space in [15], actually slightly more generally as $U_q(\mathfrak{b}_+)$.

For particular examples of this type we do not of course *need* any fancy noncommutative geometric point of view – Lie theory was already extensively developed just to handle such algebras. But if we wish to unify quantum and geometric effects then we should start taking this noncommutative geometric viewpoint even on such familiar algebras. What are 'vector fields' on $U(su_2)$? What is Fourier transform

$$\mathcal{F} : U(su_2) \to \mathbb{C}(SU_2)$$

from the momentum coordinates to the SU_2 position coordinates? These are nontrivial (but essentially solved) questions. Understanding them, we can proceed to construct more complex examples of noncommutative geometry which are neither $U(\mathfrak{g})$ nor $C(G)$, i.e. where noncommutative geometry is really needed and where both quantum and geometrical effects are unified. Vector fields, Fourier theory etc., extend to this domain and allow us to explore consistently new ideas for Planck scale physics. This approach to Planck scale physics based particularly on Fourier theory to extend the familiar $\mathbf{x}, \mathbf{p}$ reciprocity to the case of nonAbelian Lie algebras and beyond is due to the author in [2][16][3][1] [14] and elsewhere.

Notice also that the three effects exemplified by the three equations (1)–(3) are all independent. They are controlled by three different parameters $\hbar, R, m$ (say). Of course in a full theory of quantum gravity all three effects could exist together and be unified into a single noncommutative algebra generated by suitable $\mathbf{x}, \mathbf{p}$. Moreover, even if we do not know the details of the correct theory of quantum gravity, if we assume that something like Born reciprocity survives then all three effects indeed *should* show up in the effective theory where we consider almost-particle states with position and momenta $\mathbf{x}, \mathbf{p}$. It would require fine tuning or some special principle to eliminate any one of them. Also the same ideas could apply at the level of the quantum gravity field theory itself, but this is a different question.

2.2 Algebraic structure of quantum mechanics

In the above discussion we have assumed that quantum systems are described by algebras generated by position and momentum. Here we will examine this a little more closely. The physical question to keep in mind is the following: *what happens to the geometry of the classical system when you quantise?*

To see the problem consider what you obtain when you quantise a sphere or a torus. In usual quantum mechanics one takes the Hilbert space on position space, e.g. $\mathcal{H} = L^2(S^2)$ or $\mathcal{H} = L^2(T^2)$ and as 'algebra of observables' one takes $A = B(\mathcal{H})$ the algebra of all bounded (say) operators. It is decreed that every self-adjoint hermitian operator a (or its bounded exponential more precisely) is an observable of the system and its expectation value in state $|\psi> \in \mathcal{H}$ is

$$< a >_\psi = < \psi|a|\psi > .$$

The problem with this is that $B(\mathcal{H})$ *is the same algebra in all cases.* The quantum system does know about the underlying geometry of the configuration space or of the phase space in other ways; the choice of 'polarisation' on the phase space or the choice of Hamiltonian etc. – such things are generally defined using the underlying position or phase space geometry – but the abstract algebra $B(\mathcal{H})$ doesn't know about this. All separable Hilbert spaces are isomorphic (although not in any natural way) so their algebras of operators are also all isomorphic. In other words, whereas in classical mechanics we use extensively the detailed geometrical structure, such as the choice of phase space as a symplectic manifold, all of this is not recorded very directly in the quantum system. One more or less forgets it, although it resurfaces in relation to the more restricted kinds of questions (labeled by classical 'handles') one asks in practice about the quantum system. In other words, *the true quantum algebra of observables should not be the entire algebra $B(\mathcal{H})$ but some subalgebra $A \subset B(\mathcal{H})$*. The choice of this subalgebra is called the *kinematic structure* and it is precisely here that the (noncommutative) geometry of the classical and quantum system is encoded. This is somewhat analogous to the idea in geometry that every manifold can be visualised concretely embedded in some $\mathbb{R}^n$. Not knowing this and thinking that coordinates $\mathbf{x}$ were always globally defined would miss out on all physical effects that depend on topological sectors, such as the difference between spheres and tori.

Another way to put this is that by the Darboux theorem all symplectic manifolds are *locally* of the canonical form $\mathrm{d}x \wedge \mathrm{d}p$ for each coordinate. Similarly one should take (1) (which essentially generates all of $B(\mathcal{H})$, one way or another) only locally. The full geometry in the quantum system is visible only by considering more nontrivial algebras than this one to bring out the global structure. We should in fact consider all noncommutative algebras equipped with certain structures common to all quantum systems, i.e. inspired by $B(\mathcal{H})$ as some kind of local model or canonical example but not limited to it. The conditions on our algebras should also be enough to ensure that there *is* a Hilbert space around and that A can be viewed concretely as a subalgebra of operators on it.

Such a slight generalisation of quantum mechanics which allows this kinematic structure to be exhibited exists and is quite standard in mathematical physics circles. The required algebra is a *von Neumann* algebra or, for a slightly nicer theory, a C^*-algebra. This is an algebra over $\mathbb{C}$ with a $*$ operation and a norm $|| \ ||$ with certain completeness and other properties. The canonical example is $B(\mathcal{H})$ with the operator norm and $*$ the adjoint operation, and every other is a subalgebra.

Does this slight generalisation have observable consequences? Certainly. For example in quantum statistical mechanics one considers not only state vectors $|\psi>$ but 'density matrices' or generalised states. These are convex linear combinations of the projection matrices or expectations associated to state vectors $|\psi_i>$ with weights $s_i \geq 0$ and $\sum_i s_i = 1$. The expectation value in such a 'mixed state' is

$$<a>=\sum_i s_i <\psi_i|a|\psi_i> \tag{4}$$

In general these possibly-mixed states are equivalent to simply specifying the expectation directly as a linear map $<\ >$: $B(\mathcal{H}) \to \mathbb{C}$. This map respects the adjoint or $*$ operation on $B(\mathcal{H})$ so that $<a^*a>\geq 0$ for all operators a (i.e. a positive linear functional) and is also continuous with respect to the operator norm. Such positive linear functionals on $B(\mathcal{H})$ are precisely of the above form (4) given by a density matrix, so this is a complete characterisation of mixed states with reference only to the algebra $B(\mathcal{H})$, its $*$ operation and its norm. The expectations $<\ >_\psi$ associated to ordinary Hilbert space states are called the 'pure states' and are recovered as the extreme points in the topological space of positive linear functionals (i.e. those which are not the convex linear combinations of any others).

Now, if the actual algebra of observables is some subalgebra $A \subset B(\mathcal{H})$ then any positive linear functional on the latter of course restricts to one on A, i.e. defines an 'expectation state' $A \to \mathbb{C}$ which associates numbers, the expectation values, to each observable $a \in A$. But not vice-versa, i.e. the algebra A may have perfectly well-defined expectation states in this sense which are not extendable to all of $B(\mathcal{H})$ in the form (4) of a density matrix. Conversely, a pure state on $B(\mathcal{H})$ given by $|\psi>\in \mathcal{H}$ might be mixed when restricted to A. The distinction becomes crucially important for the correct analysis of quantum thermodynamic systems for example, see [17].

The analogy with classical geometry is that not every local construction may be globally defined. If one did not understand that one would miss such important things as the Bohm-Aharanov effect, for example. Although I am not an expert on the 'measurement problem' in the philosophy of quantum mechanics it does not surprise me that one would get into inconsistencies if one did not realise that the algebra of observables is a subalgebra of $B(\mathcal{H})$. And from our point of view it is precisely to understand and 'picture' the structure of the subalgebra for a given system that noncommutative geometry steps in. I would also like to add that the problem of measurement itself is a matter of matching the quantum system to macroscopic features such as the position of measuring devices. I would contend that to do this consistently one first has to know how to identify aspects of 'macrospopic structure' in the quantum system without already taking the classical limit. Only in this way can one meaningfully discuss concepts such as partial measurement or the arbitrariness of the division into measurer and measured. Such an identification is exactly the task of noncommutative geometry, which deals with extending our macroscopic intuitions and classical 'handles' over to the quantum system. Put another way, the

correspondence principle in quantum mechanics typically involves choosing local coordinates like $\mathbf{x}, \mathbf{p}$ to map over. Its refinement to correspond more of the global geometry into the quantum world is the practical task of noncommutative geometry.

2.3 Principle of representation-theoretic self-duality

With the above preambles, we are in a position to consider some speculation about Planck scale physics. Personally I believe that anything we write down that is based on our past experience and not on the deepest philosophical principles is not likely to survive except as an approximation. For example, while string theory may indeed survive to models of the Planck scale as a certain approximation valid in a certain domain, it does not have enough of a radical new philosophy to provide the true conceptual leaps. I should apologise for this belief but I do not believe that Nature cares about the historically convenient route by which we might arrive at the right concepts for the Planck scale.

So as a basis we should stick only to some of the deepest principles. In my opinion one of the deepest principles concerns the nature of mathematics itself. Namely throughout mathematics one finds an intrinsic dualism between observer and observed as follows. When we think of a function f being evaluated on $x \in X$, we could equally-well think of the same numbers as x being evaluated on f a member of some dual structure $f \in \hat{X}$:

$$\text{Result} = f(x) = x(f).$$

Such a 'turning of the tables' is a mathematical fact. For any mathematical concept X one may consider maps or 'representations' from it to some self-evident class of objects (say rational numbers or for convenience real or complex numbers) wherein our results of measurements are deemed to lie. Such representations themselves form a dual structure $\hat{X}$ of which elements of X can be equally well viewed as representations. *But is such a dual structure equally real?* I postulated in 1987 that indeed it should be so in a complete theory. Indeed[1], *The search for a complete theory of physics is the search for a self-dual formulation in the above representation-theoretic sense (The principle of representation-theoretic self-duality).* Put another way, a complete theory of physics should admit a 'polarisation' into two halves each of which is the set of representations of the other. This division should be arbitrary – one should be able to reverse interpretations (or indeed consider canonical transformations to other choices of 'polarisation' if one takes the symplectic analogy).

Note that by completeness here I do not mean knowing in more and more detail *what* is true in the real world. That consists of greater and greater complexity but it is not *theoretical* physics. I'm considering that a theorist wants to know *why* things are the way they are. Ideally I would like on my deathbed to be able to say that I have found the right point of view or theoretical-conceptual framework from which everything else follows. Working out the details of that would be far from trivial of course. This is a more or less conventional reductionist

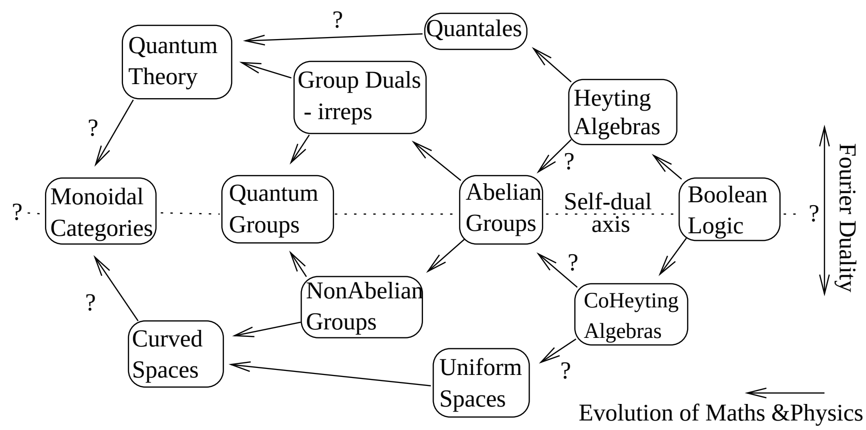

Figure 2: Representation-theoretic approach to Planck-scale physics. The unification of quantum and geometrical effects is a drive to the self-dual axis. Arrows denote inclusion functors

viewpoint except that the Principle asserts that we will not have found the required point of view unless it is self-dual.

For example, there is a sense in which geometry – or 'gravity' is dual to quantum theory or matter. This is visible for some simple models such as spheres with constant curvature where it is achieved by Fourier theory. We will be saying more about this later. If we accept this then in general terms Planck scale physics has to unify these mutually dual concepts into one self-dual structure. Ideally then Einstein's equation

$$G_{\mu\nu} = T_{\mu\nu} \tag{5}$$

would appear as some kind of *self-duality* equation within this self-dual context. Here the stress-energy tensor $T_{\mu\nu}$ measures how matter responds to the geometry, while the Einstein tensor $G_{\mu\nu}$ measures how geometry responds to matter. This is the part of Mach's principle which apparently inspired Einstein. The question is how to make these ideas precise in a representation-theoretic sense. While this still remains a long-term goal or vision, there are some toy models[3] where some of the required features can be seen. We come to them in a later section. For the moment we note only that one needs clearly some kind of noncommutative geometry because $T_{\mu\nu}$ should really be the quantum operator stress-energy and its coupling to $G_{\mu\nu}$ through its expectation value is surely only the first approximation or semiclassical limit of an operator version of (5). But an operator version of $G_{\mu\nu}$ only makes sense in the context of noncommutative geometry. What we would hope to find, in a suitable version of these ideas, is a self-dual setting where there was a dual interpretation in which $T_{\mu\nu}$ was the Einstein tensor of some dual system and $G_{\mu\nu}$ its stress-energy. In this way the duality and self-duality of the situation would be made manifest.

This is more or less where quantum groups come in, as a simple and soluble version of the more general unification problem. The situation is shown in Figure 2. Thus, the simplest theories

of physics are based on Boolean algebras (a theory consists of classification of a 'universe' set into subsets); there is a well-known duality operation interchanging a subset and its complement. The next more advanced self-dual category is that of (locally compact) Abelian groups such as $\mathbb{R}^n$. In this case the set of 1-dimensional (ir)reps is again an Abelian group, i.e. the category of such objects is self-dual. In the topological setting one has $\hat{\mathbb{R}}^n \cong \mathbb{R}^n$ so that these groups (which are at the core of linear algebra) are self-dual objects in the self-dual category of Abelian groups. Of course, Fourier theory interchanges these two. More generally, to accommodate other phenomena we step away from the self-dual axis. Thus, nonAbelian Lie groups such as SU_2 as manifolds provide the simplest examples of curved spaces. Their duals, which means constructing irreps, appear as central structures in quantum field theory (as judged by any course on particle physics in the 1960's). Wigner even defined a particle as an irrep of the Poincaré group. The unification of these two concepts, groups and groups duals was for many years an open problem in mathematics. Hopf algebras or quantum groups had been invented as the next more general self-dual category containing groups and group duals (and with Hopf algebra duality reducing to Fourier duality) back in 1947 but no general classes of quantum groups going beyond groups or group duals i.e. truly unifying the two were known. In 1986 it was possible to view this open problem as a 'toy model' or microcosm of the problem of unifying quantum theory and gravity and the bicrossproduct quantum groups such as $\mathbb{C}(G^\star) \blacktriangleright\!\!\triangleleft U(\mathfrak{g})$ were introduced on this basis as toy models of Planck scale physics[3]. The construction is self-dual (the dual is of the same general form). At about the same time, independently, some other quantum groups $U_q(\mathfrak{g})$ were being introduced from a different point of view both mathematically and physically (namely as generalised symmetries). We go into details in later sections.

We end this section with some promised philosophical remarks. First of all, why the principle of self-duality? Why such a central role for Fourier theory? The answer I believe is that something very general like this (see the introductory discussion) underlies the very nature of what it means to do science. My model (no doubt a very crude one but which I think captures some of the essence of what is going one) is as follows. Suppose that some theorist puts forward a theory in which there is an actual group G say 'in reality' (this is where physics differs from math) and some experimentalists construct tests of the theory and in so doing they routinely build representations or elements of $\hat{G}$. They will end up regarding $\hat{G}$ as 'real' and G as merely an encoding of $\hat{G}$. The two points of view are in harmony because mathematically (in a topological context)

$$G \cong \hat{\hat{G}}.$$

So far so good, but through the interaction and confusion between the experimental and theoretical points of view one will eventually have to consider both, i.e. $G \times \hat{G}$ as real. But then the theorists will come along and say that they don't like direct products, everything should interact with everything else, and will seek to unify $G, \hat{G}$ into some more complicated irreducible structure G_1, say. Then the experimentalists build $\hat{G}_1$ ... and so on. This is a kind of engine for

the evolution of Science.

For example, if one regarded, following Newton that space $\mathbb{R}^n$ is real, its representations $\hat{\mathbb{R}}^n$ are derived quantities $\mathbf{p} = m\dot{\mathbf{x}}$. But after making diverse such representations one eventually regards both $\mathbf{x}$ and $\mathbf{p}$ as equally valid, equivalent via Fourier theory. But then we seek to unify them and introduce the CCR algebra (1). And so on. Note that this is not intended to be a historical account but a theory for how things should have gone in an ideal case without the twists and turns of human ignorance.

One could consider this point of view as window dressing. Surely quantum mechanics was 'out there' and would have been discovered whatever route one took? Yes, but if if the mechanism is correct even as a hindsight, the same mechanism *does have predictive power* for the next more complicated theory. The structure of the theory of self-dual structures is nontrivial and not everything is possible. Knowing what is mathematically possible and combining with some postulates such as the above is not empty. For example, back in 1989 and motivated in the above manner it was shown that the category of monoidal categories (i.e. categories equipped with tensor products) was itself a monoidal category, i.e. that there was a construction $\hat{\mathcal{C}}$ for every such category $\mathcal{C}$[18]. Since then it has turned out that both conformal field theory and certain other quantum field theories can indeed be expressed in such categorical terms. Geometrical constructions can also be expressed categorically[19]. On the other hand, this categorical approach is still under-developed and its exact use and the exact nature of the required duality as a unification of quantum theory and gravity is still open. I would claim only 'something like that' (one should not expect too much from philosophy alone).

Another point to be made from Figure 2 is that if quantum theory and gravity already take us to very general structures such as categories themselves for the unifying concept then, in lay terms, what it means is that the required theory involves very general concepts indeed of a similar level to semiotics and linguistics (speaking about categories of categories etc.). It is almost impossible to conceive *within existing mathematics* (since it is itself founded in categories) what fundamentally more general structures would come after that. In other words, *the required mathematics is running out* it least in the manner that it was developed in this century (i.e. categorically) and at least in terms of the required higher levels of generality in which to look for self-dual structures. If the search for the ultimate theory of physics is to be restricted to logic and mathematics (which is surely what distinguishes science from, say, poetry), then this indeed correlates with our physical intuition that the unification of quantum theory and gravity is the last big unification for physics as we know it, or as were that theoretical physics as we know it is coming to an end. I would agree with this assertion except to say that the new theory will probably open up more questions which are currently considered metaphysics and make them physics, so I don't really think we will be out of a job even as theorists (and there will always be an infinite amount of 'what' work to be done even if the 'why' question was answered at some consensual level).

As well as seeking the 'end of physics', we can also ask more about its birth. Again there are many nontrivial and nonempty questions raised by the self-duality postulate. Certainly the key generalisation of Boolean logic to intuitionistic logic is to relax the axiom that $a \cup \tilde{a} = 1$ (that a or not a is true). Such an algebra is called a *Heyting algebra* and can be regarded as the birth of quantum mechanics. Dual to this is the notion of a *coHeyting algebra* in which we relax the law that $a \cap \tilde{a} = 0$. In such an algebra one can define the 'boundary' of a proposition as

$$\partial a = a \cap \tilde{a}$$

and show that it behaves like a derivation. This is surely the birth of geometry. How exactly this complementation duality extends to the Fourier duality for groups and on to the duality between more complex geometries and quantum theory is not completely understood, but there are conceptual 'physical' argument that this should be so, put forward in [1].

Briefly, in the simplest 'theories of physics' based only on logic one can work equally well with 'apples' or 'not-apples' as the names of subsets. *What happens to this complementation duality in more advanced theories of physics?* Apples curve space while not-apples do not, i.e. in physics one talks of apples as really existing while not-apples are merely an abstract concept. Clearly the self-duality is lost in a theory of gravity alone. But we have argued[1] that when one considers both gravity *and* quantum theory the self-duality can be restored. Thus when we say that a region is as full of apples as General Relativity allows (more matter simply forms a black hole which expands), which is the right hand limiting line in Figure 3[1], in the dual theory we might say that the region is as empty of not-apples as quantum theory allows, the limitation being the left slope in Figure 3. Here the uncertainty principle in the form of pair creation ensures that space cannot be totally empty of 'particles'. Although heuristic, these are arguments that quantum theory and gravity are dual and that this duality is an extension of complementation duality. Only a theory with both would be self-dual. Also, in view of a 'hole' moving in the opposite direction to a particle, the dual theory should also involves time reversal. The self-duality is something like CPT invariance but in a theory where gravitational and not only quantum effects are considered. We are proposing it as a key requirement for quantum-gravity.

2.4 Relative realism

So far we have given arguments that there is at least a correlation between the mathematical structure of self-dual structures and the progressive theories of physics from their birth in 'logic' to the projected forthcoming complete theory of everything. It should at least provide a guide to the properties that should be central in unknown theories of everything, such as what have become fashionable to call 'M-theory'.

[1] Diagrams similar to the right hand side of Figure 3 plotting the mass-energy density v size have been attributed to Brandon Carter (who was here at the conference), as a tool to plot stellar evolution

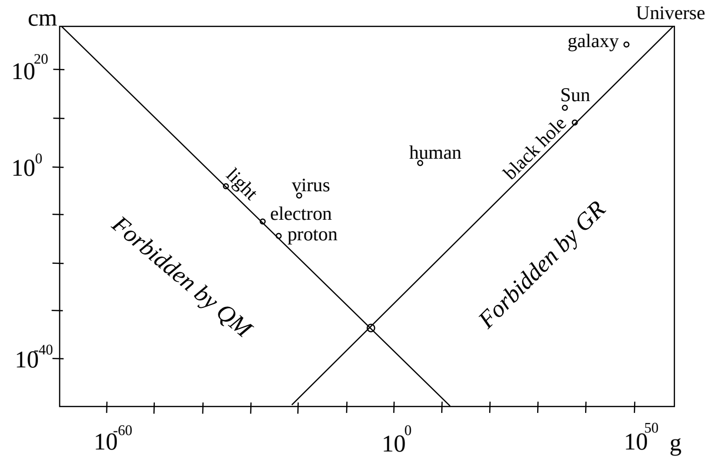

Figure 3: Range of physical phenomena, which lie in the wedge region with us in the middle. Log plots are mass-energy v size

What about going further? This section will indeed be speculative but I believe it should be considered. Suppose indeed that some mathematical-structural principles (such as the principle of representation theoretic self-duality above) could exactly pin down the ultimate theory of physics along the lines discussed. This would be like giving a list of things that we expect from a complete theory – such as renormalisability, CPT-invariance, etc., except that we are considering such general versions of these 'constraints' that they are practically what it means to be a group of people following the scientific method. If this really pins down the ultimate theory then it means that *the ultimate theory of physics is no more and no less than a self-discovery of the constraints in thinking that are taken on when one decides to look at the world as a physicist.*

If this sounds cynical it is not meant to be; it is merely a Kantian or Hegelian basis of physical reality as opposed the more conventional reductionist one that most physicists take for granted. It does not mean that physics is arbitrary or random any more than the different possible manifolds 'out there' are arbitrary. The space of all possible manifolds up to equivalence has a deep and rich structure and feels every bit as real to anyone who studies it; but it is a mathematical reality 'created' when we accept the axioms of a manifold. So what we are saying is that there is not such a fundamental difference between mathematical reality and physical reality. The main difference is that mathematicians are aware of the axioms while physicists tend to discover them 'backwards' by theorising from experience. I call this point of view *relative realism*[1]. In it, we experience reality through choices that we have forgotten about at any given moment. If we become aware of the choice the reality it creates is dissolved or 'unconstructed'. On the other hand, the reader will say that the possibility of the theory of manifolds – that the game of manifold-hunting could have been played in the first place – is itself a reality, not arbitrary. It is, but at a higher level: it is a concrete fact in a more general theory of possible

axiom systems of this type. To give another example, the reality of chess is created once we chose to play the game. If we are aware that it is a game, that reality is dissolved, but the rules of chess remain a reality although not within chess but in the space of possible board games. This gives a tree-like or hierarchical structure of reality. Reality is experienced as we look down the tree while 'awareness' or enlightenment is achieved as we look up the tree. When we are born we take on millions and millions of assumptions or rules through communication, which creates our day to day perception of reality, we then spend large parts of our lives questioning and attempting to unconstruct these assumptions as we seek understanding of the world.

Ten years ago I would have had to apologise to the reader for presenting such a philosophy or 'metamodel' of physics but, as mentioned in the Introduction, now that theories of everything are beginning to be bandied about I do believe it is time to give deeper thought to these issues. As a matter of fact the paper [1] on which most of Section 2 is based was submitted in 1987 to the Canadian *Philosophy of Science Journal* where a very enthusiastic referee conditionally accepted the paper but insisted that the arguments were basically Kantian and that I had to read Kant.[2] Kant basically said that reality was a product of human thought. From this perspective the fact that life appears somewhere near the middle of Figure 3, apart from the obvious explanation that phenomena become simpler as we approach the boundaries hence most complex in the middle so this is statistically where life would develop, has a different explanation: we created our picture of physical reality around ourselves and so not surprisingly we are near the middle.

3 Fourier theory

It is now high time to turn from philosophy to more mathematical considerations. We give more details about Fourier duality and in particular how it leads to quantum groups as a concrete 'toy model' setting to explore the above ideas. At the same time it should be clear from the general nature of the discussion above that quantum groups and even noncommutative geometry itself are only relatively simple manifestations of even more general ideas that might be approached along broadly similar lines.

First of all, usual Fourier theory on $\mathbb{R}$ is a pairing of two groups, position x and momentum p. The momentum here labels the characters on $\mathbb{R}$, i.e the elements of the dual group $\hat{\mathbb{R}}$. The corresponding character is the plane wave

$$\chi_p(x) = e^{\imath xp}$$

The group $\hat{\mathbb{R}}$ has its group structure given by pointwise multiplication

$$\chi_p \chi_{p'}(x) = \chi_p(x)\chi_{p'}(x) = \chi_{p+p'}(x)$$

[2] I duly spent the entire summer of 1989 reading up Kant and revising the paper; after which the referee rejected the paper with the immortal words 'now that the basic structure of the author's case is more exposed I do not find it clarified'!

which is therefore isomorphic to $\mathbb{R}$ as the addition of momentum. Moreover, the situation is symmetrical i.e. one could regard the same plane waves as characters $\chi_x(p)$ on momentum space. The Fourier transform is a map from functions on $\mathbb{R}$ to functions on $\hat{\mathbb{R}}$,

$$\mathcal{F}(f)(p) = \int dx f(x)\chi_p(x)$$

3.1 Loop variables and Fourier duality

It is well-known that these ideas work for any locally compact Abelian group. The local-compactness is needed for the existence of a translation-invariant measure. As physicists we can also apply these ideas formally for other groups pretending that there is such a measure. For example in [20][21][22][23] we proposed a Fourier theory approach to the quantisation of photons as follows. The elements κ of the group are disjoint unions of oriented knots (i.e. links) with a product law that consists of erasing any overlapping segments of opposite orientation. The dual group is $\mathcal{A}/\mathcal{G}$ of $U(1)$ bundles and (distributional) connections A on them. Thus given any bundle and connection, the character is the holonomy

$$\chi_A(\kappa) = e^{\imath \int_\kappa A}.$$

We considered this set-up in and the inverse Fourier transform of some well-known functions on $\mathcal{A}/\mathcal{G}$ as functions on the group of knots. For example[22],

$$\mathcal{F}^{-1}(\mathrm{CS})(\kappa) = \int \mathrm{d}A\ \mathrm{CS(A)}e^{-\imath \int_\kappa A} = e^{\frac{\imath}{2\alpha}\mathrm{link}(\kappa,\kappa)} \tag{6}$$

$$\mathcal{F}^{-1}(\mathrm{Max})(\kappa) = \int \mathrm{d}A\ \mathrm{Max(A)}e^{-\imath \int_\kappa A} = e^{\frac{\imath}{2\beta}\mathrm{ind}(\kappa,\kappa)} \tag{7}$$

where

$$\mathrm{CS}(A) = e^{\frac{\alpha\imath}{2}\int A\wedge \mathrm{d}A}, \quad \mathrm{Max}(A) = e^{\frac{\beta\imath}{2}\int {}^*\mathrm{d}A\wedge \mathrm{d}A}$$

are the Chern-Simmons and Maxwell actions, link denotes linking number, ind denotes mutual inductance.

The diagonal $\mathrm{ind}(\kappa,\kappa)$ is the *mutual self-inductance* i.e. you can literally cut the knot, put a capacitor and measure the resonant frequency to measure it. By the way, to make sense of this one has to use a wire of a finite thickness – the self-inductance has a log divergence. This is also the log-divergence of Maxwell theory when one tries to make sense of the functional integral, i.e. renormalisation has a clear physical meaning in this context[22].

Meanwhile, $\mathrm{link}(\kappa,\kappa)$ is the self-linking number[20][21][22] of a knot with itself, defined as follows. First of all, between two disjoint knots $\mathrm{link}(\kappa,\kappa')$ is the linking number as usual. We then introduce the following *regularised linking number*

$$\mathrm{link}_\epsilon(\kappa,\kappa') = \int_{||\vec{\epsilon}||<\epsilon} \mathrm{d}^3\vec{\epsilon}\ \mathrm{link}(\kappa,\kappa'_{\vec{\epsilon}})$$

where $\kappa'_{\vec{\epsilon}}$ is the knot displaced by the vector $\vec{\epsilon}$. The integrand is defined almost everywhere and hence integrable. Finally, we define the linking number as the limit of this as $\epsilon \to 0$, which is now defined even when knots touch or even on the same knot. At the time of [20][21][22], actually back in 1986, I made the following conjecture which is still open.

Conjecture 1 *Intersections that are worse and worse (i.e. so that higher and higher derivatives coincide at the point of intersection) contribute fractions with greater and greater denominators to the regularised linking number, but the linking number remains in $\mathbb{Q}$. In the extreme limit of total overlap the self-linking number is a generic element of $\mathbb{R}$.*

As evidence, if the knots intersect transversally then it is easy to see that one obtains for the regularised linking an integer $\pm\frac{1}{2}$. This is just because half the displacements will move one knot in to link more with the other, and the other half to unlink. [3] Although the conjecture remains open, it does appear that it could be interesting for loop variable quantum gravity where it would imply certain rationality properties. By the way, one might need to average over infinitesimal rotations as well as displacements to prove it.

Note also that our point of view in [20][21][22] was *distributional* because as well as considering honest smooth connections we considered 'connections' defined entirely by their holonomy. In particular, given a knot κ we defined the distribution A_κ by its character as

$$e^{\imath \int_{\kappa'} A_\kappa} = e^{\imath \mathrm{link}(\kappa,\kappa')}.$$

Such distributions are quite interesting. For example[21][23] if one formally evaluates the Maxwell action in these one has[20]

$$\mathrm{Max}(A_\kappa) = e^{\frac{\imath}{4\beta}\delta^2(0)\int_\kappa \mathrm{d}t\dot{\kappa}\cdot\dot{\kappa}}, \tag{8}$$

the Polyakov string action. In other words, string theory can be embedded into Maxwell theory by constraining the functional integral to such 'vortex' configurations. An additional Chern-Simons term becomes similarly a 'topological mass term' $\mathrm{link}(\kappa,\kappa)$ that we proposed to be added to the Polyakov action.

Finally, these ideas also have analogues in the Hamiltonian formulation. Thus the CCR's for the gauge field can be equivalently formulated as

$$[\int_\kappa A, \int_\Sigma E] = 4\pi\imath\alpha \mathrm{link}(\kappa,\partial\Sigma)$$

which is a signed sum of the points of intersection of the loop with the surface. This is the point of view by which loop variables were introduced in physics in the 1970's (as an approach to QCD

[3]This result for transverse intersections, the regularised linking itself from the conjecture for higher intersections were shown to Abbay Ashtekar (and Lou Kauffman) during the ICAMP meeting in Swansea 1988 in advance of the eventual publication in [22].

on lattices) by Mandelstam and others. We have observed in [20] that this has an interpretation as noncommutative geometry, generalising the noncommutative torus $v^n u^m = e^{\imath\alpha mn} u^m v^n$ to

$$v_\kappa u_{\kappa'} = e^{4\pi\imath\alpha \mathrm{link}(\kappa,\kappa')} u_{\kappa'} v_\kappa \tag{9}$$

where integers are replaced by knots or links. Here the physical picture is

$$u_\kappa = e^{\imath \int_\kappa A}, \quad v_\kappa = e^{\imath \int_\kappa \tilde{A}} \tag{10}$$

where $\tilde{A}$ is a dual connection such that $E = \mathrm{d}\tilde{A}$. So constructing the u, v is equivalent to constructing some distributional operators A, E with the usual CCR's. This point of view from [20][21] was eventually published in [23] as a noncommutative-geometric approach to the quantisation of photons.

It is also an interesting question how all of these ideas generalise from $U(1)$ to nonAbelian groups. Thus, in place of the Abelian group of knots one can first of all consider some kind of nonAbelian group of parameterized loops in the manifold, i.e. maps rather than the images of these maps. (The inequivalent classes of elements in this are the fundamental group π_1 of the manifold.) This should be paired via the Wilson loop or holonomy with nonAbelian bundles and connections. The precise groups and their duality here is a little hazy but one should think of this roughly speaking as what goes on in the construction of knot invariants from the WZW model (or from quantum group). Thus one could argue[21][22] that the relationship between the Jones polynomial J and SU_2-Chern-Simons theory should be viewed as *some kind of nonAbelian Fourier transform*

$$\mathcal{F}^{-1}(\mathrm{CS}_{SU_2})(\kappa) \sim e^{J(\kappa)} \tag{11}$$

with the Jones polynomial in the role of self-linking number[4]. We will discuss Fourier transform on nonAbelian groups in the next section using quantum group methods, though I should say that it still remains to make (11) precise along such lines. The reformulation of quantum group invariants as Vassiliev invariants and the Kontsevich integrals (which generalise the linking number) could be viewed, however, as a perturbative step in this direction.

It does seems that many of these ideas have emerged in modern times in the loop variable approach to quantum gravity[24][25], with the nonAbelian group SU_2 (or another group) in place of $U(1)$. However, I want to close this section with some ideas in this area that I still did not see emerge. Indeed, what the loop variable approach tells us is that the gravitational field when recast as a spin connection is in some sense the conjugate variable to something of manifest topological and diffeomorphism-invariant meaning – knots and links in the manifold. In the same spirit it is obvious that scaler fields correspond to points in the manifold[21][23]. What about in the other direction? I would conjecture that there is another field or force in

[4]This conjecture dates from 1986 at the time of [20] but was not published until [22], following Witten's discovery of the relation between the WZW model and the Jones polynomial at the ICAMP in Swansea in 1988

nature (possibly as yet undiscovered) corresponding to surfaces rather than loops (and so on). Then just as gauge fields tend to detect π_1, the new field would for example detect π_2. Note that in the $U(1)$ case the pairing of surfaces is of course with 2-forms (and the 2nd cohomology is the Abelianisation of π_2) – we would need a nonAbelian version of that.

Actually this conjecture was one of my main motivations back in 1986 in the slightly different context of a search for such Fourier transform or 'surface transport' methods for QCD. First of all, one can ask: if the Fourier transform of the nonAbelian Chern-Simons theory gives the quantum group link invariants as in (11), *what is the Fourier transform of the Yang-Mills action?* According to (7) it should be some kind of some kind of 'nonAbelian self-inductance'. The extra ingredient in QCD is of course confinement. Related to this is the need for some kind of 'nonAbelian Stokes theorem'. While no continuum version of the latter exists, let us suppose that is has somehow been defined, i.e. the Lie group G-valued 'parallel transport' of a nonAbelian Lie-algebra valued 2-form F over a surface such that if F is the curvature of a gauge field then

$$e^{\imath\int_\Sigma F} = e^{\imath\int_{\partial\Sigma} A}. \tag{12}$$

While this is not really possible (except rather artificially on a lattice by specifying paths parallel transporting back to a fixed based point) we suppose something *like* this.

Conjecture 2 *With such a nonAbelian surface transport, the QCD vacuum expectation value of the flux of the quantized curvature F through a closed surface is an invariant of the surface.*

The point is that one usually considers only planar spans of loops in QCD and Wilson's criterion for confinement says that these are area law. On the other hand if one considered a small planar loop spanned by a large surface 'ballooning out' from the loop one would still expect some finite result (since a large area), but on the other hand the boundary curve itself could be shrunk to zero so that its planar spanning surface also shrinks to zero and Wilson's criterion would give 1. The conjecture is that these two effects cancel out and one has in fact something that depends only on the topological class of the surface. This does require, however, making sense of (12) which might require some accompanying new fields. On the other hand, at least one standard objection to the above ideas *was* solved, namely we do not need to take traces of the holonomies etc., which means that we are considering the expectations of gauge-non-invariant operators. It was argued in [26] that one could do this in the context of a version of the background field method. This is important because one can then analyse and prove confinement locally as the statement that the expectation $< F >$ is a (nonAbelian) curvature + a non-curvature part (the latter was shown in [26] to be the skew-symmetrized gluon two-point function). The first part is 'perimeter law' and the second is 'area law' and corresponds to confinement infinitesimally. The conjecture would extend these ideas globally. At the end of the day, however, the strong force itself might emerge as related to surfaces in much the same way as gravity is to loops via the loop gravity and spin connection formalisms.

3.2 NonAbelian Fourier Transform

To generalise Fourier theory beyond Abelian groups we really have to pass to the next more general self-dual category, which is that of Hopf algebras or quantum groups. A Hopf algebra is

- A unital algebra $H, 1$ over the field $\mathbb{C}$ (say)
- A coproduct $\Delta : H \to H \otimes H$ and counit $\epsilon : H \to \mathbb{C}$ forming a *coalgebra*, with Δ, ϵ algebra homomorphisms.
- An antipode $S : H \to H$ such that $\cdot(S \otimes \mathrm{id})\Delta = 1\epsilon = \cdot(\mathrm{id} \otimes S)\Delta$.

Here a coalgebra is just like an algebra but with the axioms written as maps and arrows on the maps reversed. Thus coassociativity means

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \tag{13}$$

etc. The axioms mean that the adjoint maps $\Delta^* : H^* \otimes H^* \to H^*$ and $\epsilon^* : \mathbb{C} \to H^*$ make H^* into an algebra. Here ϵ^* is simply ϵ regarded as an element of H^*. The meaning of the antipode S is harder to explain but it generalises the notion of inverse. It is a kind of 'linearised inversion'.

For a Hopf algebra, at least in the finite-dimensional case (i.e. with a suitable definition of dual space in general) the axioms are such that H^* is again a Hopf algebra. Its coproduct is the adjoint of the product of H and its counit is the unit of H regarded as a map on H^*. This is why the category of Hopf algebras is a self-dual one. For more details we refer to [6].

We will give examples in a moment, but basically these axioms are set up to define Fourier theory. Thinking of H as like 'functions on a group', the coproduct corresponds to the group product law by dualisation. Hence a translation-invariant integral means in general a map $\int : H \to \mathbb{C}$ such that

$$(\int \otimes \mathrm{id})\Delta = 1 \int \tag{14}$$

Meanwhile, the notion of plane wave or exponential should be replaced by the canonical element

$$\exp = \sum_a e_a \otimes f^a \in H \otimes H^* \tag{15}$$

where $\{e_a\}$ is a basis and $\{f^a\}$ is a dual basis. We can then define Fourier transform as

$$\mathcal{F} : H \to H^*, \quad \mathcal{F}(h) = \int (\exp)h = (\int \sum_a e_a h) f^a. \tag{16}$$

There is a similar formula for the inverse $H^* \to H$.

The best way to justify all this is to see how it works on our basic example for Fourier theory. Thus, we take $H = \mathbb{C}[x]$ the algebra of polynomials in one variable, as the coordinate algebra of $\mathbb{R}$. It forms a Hopf algebra with

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \epsilon x = 0 \quad Sx = -x \tag{17}$$

as an expression of the additive group structure on $\mathbb{R}$. Similarly we take $\mathbb{C}[p]$ for the coordinate algebra of another copy of $\mathbb{R}$ with generator p dual to x (the additive group $\mathbb{R}$ is self-dual).

Example 1 *The Hopf algebras $H = \mathbb{C}[x]$ and $H^* = \mathbb{C}[p]$ are dual to each other with $\langle x^n, p^m \rangle = (-\imath)^n \delta_{n,m} n!$ (under which the coproduct of one is dual to the product of the other). The exp element and Fourier transform is therefore*

$$\exp = \sum \imath^n \frac{x^n \otimes p^n}{n!} = e^{\imath x \otimes p}, \quad \mathcal{F}(f)(p) = \int_{-\infty}^{\infty} \mathrm{d}x f(x) e^{\imath x \otimes p}.$$

Apart from an implicit $\otimes$ symbol which one does not usually write, we recover usual Fourier theory. Both the notion of duality and the exponential series are being treated a bit formally but can be made precise.

Let is now apply this formalism to Fourier theory on classical but nonAbelian groups. We use Hopf algebra methods because Hopf algebras include both groups and group duals even in the nonAbelian case, as we have promised in Section 2. Thus, if $\mathfrak{g}$ is a Lie algebra with associated Lie group G, we have two Hopf algebras, dual to each other. One is $U(\mathfrak{g})$ the enveloping algebra with

$$\Delta \xi = \xi \otimes 1 + 1 \otimes \xi, \quad \xi \in \mathfrak{g}$$

and the other is the algebra of coordinate functions $\mathbb{C}(G)$. If G is a matrix group the functions t_{ij} which assign to a group element its ij matrix entry generate the coordinate algebra. Of course, they commute i.e. $\mathbb{C}(G)$ is the commutative polynomials in the $t^i{}_j$ modulo some other relations that characterise the group. Their coproduct is

$$\Delta t^i{}_j = t^i{}_k \otimes t^k{}_j$$

corresponding to the matrix multiplication or group law. The pairing is

$$\langle t^i{}_j, \xi \rangle = \rho(\xi)^i{}_j$$

where ρ is the corresponding matrix representation of the Lie algebra. The canonical element or exp is given by choosing a basis for $U(\mathfrak{g})$ and finding its dual basis.

Example 2 *$H = \mathbb{C}(SU_2) = \mathbb{C}[a, b, c, d]$ modulo the relation $ad - bc = 1$ (and unitarity properties). It has coproduct*

$$\Delta a = a \otimes a + b \otimes c, \quad \text{etc.,} \quad \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is dually paired with $H^* = U(su_2)$ in its antihermitian usual generators $\{e_i\}$ with pairing

$$\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, e_i \rangle = \frac{\imath}{2} \sigma_i,$$

defined by the Pauli matrices. Let $\{e_1^a e_2^b e_3^c\}$ be a basis of $U(su_2)$ and $\{f^{a,b,c}\}$ the dual basis. Then

$$\exp = \sum_{a,b,c} f^{a,b,c} \otimes e_1^a e_2^b e_3^c \in \mathbb{C}(SU_2)\bar{\otimes} U(su_2)$$

$$\mathcal{F}(f) = \int_{SU_2} \mathrm{d}u f(u) f^{a,b,c}(u) \otimes e_1^a e_2^b e_3^c.$$

Here $\mathrm{d}u$ denotes the right-invariant Haar measure on SU_2. For a geometric picture one should think of e_i as noncommuting coordinates i.e. regard $U(su_2)$ as a 'noncommutative space' as in (3). An even simpler example is the Lie algebra $\mathfrak{b}_+$ with generators x, t and relations $[x, t] = \imath\lambda x$. Its enveloping algebra could be viewed as a noncommutative analogue of 1+1 dimensional space-time.

Example 3 *c.f. [27] The group B_+ of matrices of the form*

$$\begin{pmatrix} e^{\lambda\omega} & k \\ 0 & 1 \end{pmatrix}$$

has coordinate algebra $\mathbb{C}(B_+) = \mathbb{C}[k, \omega]$ with coproduct

$$\Delta e^{\lambda\omega} = e^{\lambda\omega} \otimes e^{\lambda\omega}, \quad \Delta k = k \otimes 1 + e^{\lambda\omega} \otimes k$$

Its duality pairing with $U(\mathfrak{b}_+)$ is generated by $\langle x, k\rangle = -\imath, \langle t, \omega\rangle = -\imath$ and the resulting exp and Fourier transform are

$$\exp = e^{\imath k\omega} e^{\imath\omega t}, \quad \mathcal{F}(: f(x,t) :) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}t \; e^{\imath kx} e^{\imath\omega t} f(e^{\lambda\omega} x, t)$$

where $: f(x,t) : \in U(\mathfrak{b}_+)$ by normal ordering x to the left of t.

Similarly (putting a vector $\vec{x}$ in place of x) the algebra $[\vec{x}, t] = \imath\lambda\vec{x}$ is merely the enveloping algebra of the Lie algebra of the group $\mathbb{R}\bowtie\mathbb{R}^n$ introduced (for $n = 2$) in [28] and could be viewed as some kind of noncommutative space-time in $1 + n$ dimensions. This was justified in 1+3 dimensions in [29], where it was shown to be the correct 'kappa-deformed' Minkowski space covariant under a 'kappa-deformed' Poincaré quantum group which had been proposed earlier[30]. We see that Fourier transform then connects it to the classical coordinate algebra $\mathbb{C}(\mathbb{R}\bowtie\mathbb{R}^n)$ of the nonAbelian group $\mathbb{R}\bowtie\mathbb{R}^n$, this time with commuting coordinates $(\vec{k}, \omega)$. This demonstrates in detail what we promised that noncommutavity of spacetime is related under Fourier transform to nonAbelianness (which typically means curvature) of the momentum group. Under Fourier theory it means that *all* noncommutative geometrical constructions and problems on this spacetime can be mapped over and solved as classical geometrical constructions on the nonAbelian momentum space.

This Fourier transform approach was demonstrated recently in [31], where we analyse the gamma-ray burst experiments mentioned in Giovanni Amelino-Camelia's lectures at the conference, from this point of view. In contrast to previous suggestions[32] (based on the deformed Poincaré algebra) we are able to justify the dispersion relation

$$\lambda^{-2}\left(e^{\lambda\omega}+e^{-\lambda\omega}-2\right)-\vec{k}^2 e^{-\lambda\omega}=m^2 \tag{18}$$

as a well-defined mass-shell in the classical momentum group $\mathbb{R}{\triangleright\!\!<}\mathbb{R}^3$ and give some arguments that the plane waves being of the form $e^{\imath\vec{k}\cdot\vec{x}}e^{\imath\omega t}$ above would have wave velocities given by $v_i=\frac{\partial\omega}{\partial k_i}$ (no meaningful justification for this of any kind had been given before). In particular, one has a variation in arrival time for a gamma ray emitted a distance L away

$$\delta T=\lambda L\delta k=\lambda\frac{L}{c}\delta\omega \tag{19}$$

as one varies the momentum by δk or the energy by $\delta\omega$. Apparently such theoretical predictions can actually be measured for gamma ray bursts that travel cosmological distances. Of course, one needs to know the distance L and use the predicted L-dependence to filter out other effects and also to filter out our lack of knowledge of the initial spectrum of the bursts. It is also conjectured in [31] that the nonAbelianness of the momentum group shows up as CPT violation and might be detected by ongoing neutral-kaon system experiments. Of course, there is nothing stopping one doing field theory in the form of Feynman rules on our classical momentum group either, except that one has to make sense of the meaning of nonAbelianess in the addition of momentum. As explained in Section 2 one can use similar techniques to those for working on curved position space, but now in momentum space, i.e. I would personally call such effects, if detected, 'cogravity'. The idea is that quantum gravity should lead to both gravitational and these more novel cogravitational effects at the macroscopic level.

Let us note finally that these nonAbelian Fourier transform ideas also work fine for finite groups and could be useful for crystallography.

4 Bicrossproduct model of Planck-scale physics

So far we have only really considered groups or their duals, albeit nonAbelian ones. The whole point of Hopf algebras, however, is that there exist examples going truly beyond these but with many of the same features, i.e. with properties of groups and group duals unified. It is high time to give some examples of Hopf algebras going beyond groups and group duals i.e. neither commutative like $\mathbb{C}(G)$ not the dual concept (cocommutative) like $U(\mathfrak{g})$, i.e. *genuine* quantum groups.

We recall from Section 2 that the unification of groups and group duals is a kind of microcosm or 'toy model' of the problem of unifying quantum theory and gravity. So our first class of quantum groups (the other to be described in a later section) come from precisely this point of view.

4.1 The Planck-scale quantum group

By 'toy model' we mean of course some kind of effective theory with stripped-down degrees of freedom but incorporating the idea that Planck scale effects would show up when we try to unify quantum mechanics and geometry through noncommutative geometry. But actually our approach can make a much stronger statement than this: we envisage that the model appears as some effective limit of an unknown theory of quantum-gravity which to lowest order would appear as spacetime and conventional mechanics on it – but even if the theory is unknown we can use the intrinsic structure of noncommutative algebras to classify *a priori* different possibilities. This is much as a phenomenologist might use knowledge of topology or cohomology to classify different *a priori* possible effective Lagrangians.

Specifically, if H_1, H_2 are two quantum groups there is a theory of the space $\mathrm{Ext}_0(H_1, H_2)$ of possible extensions

$$0 \to H_1 \to E \to H_2 \to 0$$

by some Hopf algebra E obeying certain conditions. We do not need to go into the mathematical details here but in general one can show that $E = H_1 \blacktriangleright\!\!\triangleleft H_2$ a 'bicrossproduct' Hopf algebra. Suffice it to say that the conditions are 'self-dual' i.e. the dual of the above extension gives

$$0 \to H_2^* \to E^* \to H_1^* \to 0$$

as another extension dual to the first, in keeping with our philosophy of self-duality of the category in which we work. We also note that by Ext_0 we mean quite strong extensions. There is also a weaker notion that admits the possibilities of cocycles as well, which we are excluding, i.e. this is only the topologically trivial sector in a certain nonAbelian cohomology.

Theorem 1 *[3][16]* $\mathrm{Ext}_0(\mathbb{C}[x], \mathbb{C}[p]) = \mathbb{R}\hbar \oplus \mathbb{R}\mathsf{G}$*, i.e. the different extensions*

$$0 \to \mathbb{C}[x] \to ? \to \mathbb{C}[p] \to 0$$

of position $\mathbb{C}[x]$ by momentum $\mathbb{C}[p]$ forming a Hopf algebra are classified by two parameters which we denote $\hbar, \mathsf{G}$ and take the form

$$? \cong \mathbb{C}[x] \blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}} \mathbb{C}[p].$$

Explicitly this 2-parameter Hopf algebra is generated by x, p with the relations and coproduct

$$[x, p] = \imath\hbar(1 - e^{-\frac{x}{\mathsf{G}}}), \quad \Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta p = p \otimes e^{-\frac{x}{\mathsf{G}}} + 1 \otimes p.$$

This is called the *Planck scale quantum group*. It is a bit more than just some randomly chosen deformation of the coordinate algebra of the usual group $\mathbb{R}^2$ of phase space of a particle in one dimension: in physical terms what we are saying is that if we are given $\mathbb{C}[x]$ the position

coordinate algebra and $\mathbb{C}[p]$ defined *a priori* as the natural momentum coordinate algebra then *all possible* quantum phases spaces built from x, p in a controlled way that preserves duality ideas (Born reciprocity) and retains the group structure of classical phase space as a quantum group are of this form labeled by two parameters $\hbar, \mathsf{G}$. We have not put these parameters in by hand – they are simply the mathematical possibilities being thrown at us. *In effect we are showing how one is forced to discover both quantum and gravitational effects from certain structural self-duality considerations.*

The only physical input here is to chose suggestive names for the two parameters by looking at limiting cases. We also should say what we mean by 'natural momentum coordinate'. What we mean is that the interpretation of p should be fixed before hand, e.g. we stipulate before hand that the Hamiltonian is $h = p^2/2m$ for a particle on our quantum phase space. Then the different commutation relations thrown up by the mathematical structure imply different dynamics. If one wants to be more conventional then one can define $\tilde{p} = p(1 - e^{-\frac{x}{\mathsf{G}}})^{-1}$ with canonical commutation relations but some nonstandard Hamiltonian,

$$[x, \tilde{p}] = \imath\hbar, \quad h = \frac{\tilde{p}^2}{2m}(1 - e^{-\frac{x}{\mathsf{G}}})^2.$$

Thus our approach is slightly unconventional but is motivated rather by the strong principle of equivalence that from some point of view the particle should be free. We specify x, p before-hand to be in that frame of reference and then explore their possible commutation relations. Of course the theorem can be applied in other contexts too whenever the meaning of x, p is fixed before hand, perhaps by other criteria.

4.1.1 The quantum flat space $\mathsf{G} \to 0$ limit

Clearly in the domain where x can be treated as having values > 0, i.e. for a certain class of quantum states where the particle is confined to this region, we clearly have flat space quantum mechanics $[x, p] = \imath\hbar$ in the limit $\mathsf{G} \to 0$.

4.1.2 The classical $\hbar \to 0$ limit

On the other hand, as $\hbar \to 0$ we just have the commutative polynomial algebra $\mathbb{C}[x, p]$ with the coalgebra shown. This is the coordinate algebra of the group B_- of matrices of the form

$$\begin{pmatrix} e^{-\frac{x}{\mathsf{G}}} & 0 \\ p & 1 \end{pmatrix}$$

which is therefore the classical phase space for general G of the system.

4.1.3 The dynamics

The meaning of the parameter G can be identified, at least roughly, as follows. In fact the meaning of p mathematically in the construction is that it acts on the position $\mathbb{R}$ inducing a

flow. For such dynamical systems the Hamiltonian is indeed naturally $h = p^2/m$ and implies that

$$\dot{p} = 0, \quad \dot{x} = \frac{p}{m}\left(1 - e^{-\frac{x}{\mathsf{G}}}\right) + O(\hbar) = v_\infty\left(1 - \frac{1}{1 + \frac{x}{\mathsf{G}} + \cdots}\right) + O(\hbar)$$

where we identify p/m to $O(\hbar)$ as the velocity $v_\infty < 0$ at $x = \infty$. We see that as the particle approaches the origin it goes more and more slowly and in fact takes an infinite amount of time to reach the origin. Compare with the formula in standard radial infalling coordinates

$$\dot{x} = v_\infty\left(1 - \frac{1}{1 + \frac{1}{2}\frac{x}{\mathsf{G}}}\right)$$

for the distance from the event horizon of a Schwarzschild black hole with

$$\mathsf{G} = \frac{G_{\rm Newton}M}{c^2},$$

where M is the background gravitational mass and c is the speed of light. Thus the heuristic meaning of G in our model is that it measures the background mass or radius of curvature of the classical geometry of which our Planck scale Hopf algebra is a quantisation.

These arguments are from [3]. Working a little harder, one finds that the quantum mechanical limit is valid (the effects of G do not show up within one Compton wavelength) if

$$mM << m_{\rm Planck}^2,$$

while the curved classical limit is valid if

$$mM >> m_{\rm Planck}^2.$$

See also [6]. *The Planck-scale quantum group therefore truly unifies quantum effects and 'gravitational' effects in the context of Figure 3.*

Of course our model is only a toy model and one cannot draw too many conclusions given that our treatment is not even relativistic. The similarity to the Schwarzschild black-hole is, however, quite striking and one could envisage more complex examples which hit that exactly on the nose. The best we can say at the moment is that the search to unify quantum theory and gravity using such methods leads to tight constraints and features such as event-horizon-like coordinate singularities. Theorem 1 says that it is not possible to make a Hopf algebra for x, p with the correct classical limit in this context without such a coordinate singularity.

4.1.4 The quantum-gravity $\hbar, \mathsf{G} \to \infty, \frac{\mathsf{G}}{\hbar} = \lambda$ limit

Having analysed the two familiar limits we can consider other 'deep quantum-gravity' limits. For example sending both our constants to ∞ but preserving their ratio we have

$$[x, p] = \imath\lambda x, \quad \Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta p \otimes 1 + 1 \otimes p$$

which is once again $U(\mathfrak{b}_+)$ regarded as in Example 3 in Section 3 'up side down' as a quantum space. The higher-dimensional analogues are 'κ-deformed' Minkowski space[29] as explained in Section 3, i.e. the Planck-scale quantum group puts some flesh on the idea that this might indeed come out of quantum gravity as some kind of effective limit[27]. Time itself would have to appear as $t = p$, (or $t = \sum_i p_i$ for the higher dimensional analogues) in this limit from the momenta conjugate in the effective quantum gravity theory to the position coordinates. This speculative possibility is discussed further in [31]. At any rate this deformed Minkowski space is at least mathematically nothing but a special limit of the Planck-scale quantum group from [3]. It gives some idea how the self-duality ideas of Section 2 might ultimately connect to testable predictions for Planck scale physics e.g. testable by gamma-ray bursts of cosmological origin.

4.1.5 The algebraic structure and Mach's Principle

The notation $\mathbb{C}[x]\blacktriangleright\!\!\triangleleft\mathbb{C}[p]$ for the Planck-scale quantum group reflects its algebraic structure. As an algebra it is a cross product $\mathbb{C}[x]\rtimes\mathbb{C}[p]$ by the action $\triangleright$ of $\mathbb{C}[p]$ on $\mathbb{C}[x]$ by

$$p\triangleright f(x) = -\imath\hbar(1 - e^{-\frac{x}{G}})\frac{\partial}{\partial x}f \tag{20}$$

which means that it is a more or less standard 'Mackey quantisation' as a dynamical system. It can also be viewed as the deformation-quantization of a certain Poisson bracket structure on $\mathbb{C}(B_-)$ if one prefers that point of view. On the other hand its coproduct is obtained in a similar but dual way as a semidirect coproduct $\mathbb{C}[x]\blacktriangleright\!\!<\mathbb{C}[p]$ by a coaction of $\mathbb{C}[x]$ on $\mathbb{C}[p]$. This coaction is induced by an action of x on functions $f(p)$ of similar form to the above but with the roles reversed. In other words, *matching the action of momentum on position is an 'equal and opposite' coaction of position back on momentum.* This is indeed inspired by the ideas of Mach[16] as was promised in Section 2.

4.1.6 Observable-state duality and T-duality

The phrase 'equal and opposite' has a precise consequence here. Namely the algebra corresponding to the coalgebra by dualisation has a similar cross product form by an analogous action of x on p. More precisely, one can show that

$$(\mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,G}\mathbb{C}[p])^*\cong\mathbb{C}[\bar{p}]\triangleright\!\!\blacktriangleleft_{\frac{1}{\hbar},\frac{G}{\hbar}}\mathbb{C}[\bar{x}], \tag{21}$$

where $\mathbb{C}[p]^* = \mathbb{C}[\bar{x}]$ and $\mathbb{C}[x]^* = \mathbb{C}[\bar{p}]$ in the sense of an algebraic pairing as in Example 1 in Section 3. Here $\langle p, \bar{x}\rangle = \imath$ etc., which then requires a change of the parameters as shown to make the identification precise. So the Planck-scale quantum group is self-dual up to change of parameters.

This means that whereas we would look for observables $a \in \mathbb{C}[x]\blacktriangleright\!\!\triangleleft\mathbb{C}[p]$ as the algebra of observables and states $\phi \in \mathbb{C}[\bar{p}]\triangleright\!\!\blacktriangleleft\mathbb{C}[\bar{x}]$ as the dual linear space, with $\phi(a)$ the expectation of a

in state ϕ (See section 2.2), there is a dual interpretation whereby

$$\text{Expectation} = \phi(a) = a(\phi)$$

for the expectation of ϕ in 'state' a with $\mathbb{C}[\bar{p}]\triangleright\!\!\blacktriangleleft\mathbb{C}[\bar{x}]$ the algebra of observables in the dual theory. More precisely, only self-adjoint elements of the algebra are observables and positive functionals states, an a state ϕ will not be exactly hermitian in the dual theory etc. But the physical hermitian elements in the dual theory will be given by combinations of such states, and vice versa. This is a concrete example of observable-state duality as promised in Section 2. It was introduced by the author in [3].

Also conjectured at the time of [3] was that this duality should be related to T-duality in string theory. As evidence is the inversion of the constant $\hbar$. In general terms coupling inversions are indicative of such dualities. Notice also that Fourier transform implements this T-duality-like transformation as

$$\mathcal{F} : \mathbb{C}[x]\blacktriangleright\!\!\triangleleft_{\hbar,\mathsf{G}}\mathbb{C}[p] \to \mathbb{C}[\bar{p}]\triangleright\!\!\blacktriangleleft_{\frac{1}{\hbar},\frac{\mathsf{G}}{\hbar}}\mathbb{C}[\bar{x}]$$

Explicitly, it comes out as[27]

$$\mathcal{F}(:f(x,p):) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}p\; e^{-\imath(\bar{p}+\frac{\imath}{\mathsf{G}})x} e^{-\imath\bar{x}(p+p\triangleright)} f(x,p), \tag{22}$$

where $\triangleright$ is the action (20) and $f(x,p)$ is a classical function considered as defining an element of the Planck-scale quantum group by normal ordering x to the left.

The duality here is not exactly T-duality in string theory but has some features like it. On the other hand it is done here at the quantum level and not in terms only of Lagrangians. In this sense the observable state duality can give an idea about what should be 'M-theory' in string theory. Thus, at the moment all that one knows really is that the conjectured M-theory should be some form of algebraic structure with the property that it has different semiclassical limits with different Lagrangians related to each other by S,T dualities (etc.) at the classical level. Our observable-state duality ideas [3][1][19] as well as more recent work on T-duality suggests that:

Conjecture 3 *M-theory should be some kind of algebraic structure possessing one or more dualities in a representation-theoretic or observable-state sense.*

Actually there is an interesting anecdote here. I once had a chance to explain the algebraic duality ideas of my PhD thesis to Edward Witten at a reception in MIT in 1988 after his colloquium talk at Harvard on the state of string theory. He asked me 'is there a Lagrangian' and when I said 'No, it is all algebraic; classical mechanics only emerges in the limits, but there are two different limits related by duality', Witten rightly (at the time) gave me a short lecture about the need for a Lagrangian. 9 years later I was visiting Harvard and Witten gave a similarly-titled colloquium talk on the state of string theory. He began by stating that there was some algebraic structure called M-theory with Lagrangians appearing only in different limits.

4.1.7 The noncommutative differential geometry

The lack of Lagrangians and other familiar structures in the full Planck-scale theory was certainly a valid criticism back in 1988. Since then, however, noncommutative geometry has come a long way and one is able to 'follow' the geometry as we quantise the system using these modern techniques. We do not have the space to recall the whole framework but exterior algebras, partial derivatives etc., make sense for quantum groups and many other noncommutative geometries. For the Planck-scale quantum group one has[27],

$$\partial_p : f(x,p) := \frac{\mathsf{G}}{\imath\hbar} : (f(x,p) - f(x,p-\imath\frac{\hbar}{\mathsf{G}})) :, \tag{23}$$

$$\partial_x : f(x,p) :=: \frac{\partial}{\partial x} f : - \frac{p}{\mathsf{G}} \partial_p : f : \tag{24}$$

which shows the effects of $\hbar$ in modifying the geometry. Differentiation in the p direction becomes 'lattice regularised' albeit a little strangely with an imaginary displacement. In the deformed-Minkowski space setting where $p = t$ it means that the Euclidean version of the theory is related to the Minkowski one by a Wick-rotation is being lattice-regularised by the effects of $\hbar$.

Also note that for fixed $\hbar$ the geometrical picture blows up when $\mathsf{G} \to 0$. I.e the usual flat space quantum mechanics CCR algebra does not admit a deformation of conventional differential calculus on $\mathbb{R}^2$ – *one needs a small amount of 'gravity' to be present for a geometrical picture in the quantum theory.* This is also evident in the exterior algebra[27]

$$f\mathrm{d}x = (\mathrm{d}x)f, \quad f\mathrm{d}p = (\mathrm{d}p)f + \frac{\imath\hbar}{\mathsf{G}}\mathrm{d}f$$

for the relations between 'functions' f in the Planck-scale quantum group and differentials. The higher exterior algebra looks more innocent with

$$\mathrm{d}x \wedge \mathrm{d}x = 0, \quad \mathrm{d}x \wedge \mathrm{d}p = -\mathrm{d}p \wedge \mathrm{d}x, \quad \mathrm{d}p \wedge \mathrm{d}p = 0 \tag{25}$$

Starting with the differential forms and derivatives, one can proceed to gauge theory, Riemannian structures etc., in some generality. One can also write down 'quantum' Poisson brackets and Hamiltonians[27] and (in principle) Lagrangians in the full noncommutative theory. Such tools should help to bridge the gap between model building via classical Lagrangians, which I personally do not think can succeed at the Planck scale, and some of the more noncommutative-algebraic ideas in Section 2.

4.2 Higher dimensional analogue

The Planck-scale quantum group is but the simplest in a family of quantum groups with similar features and parameters. We work from now with $\mathsf{G} = \hbar = 1$ for simplicity but one can always put the parameters back.

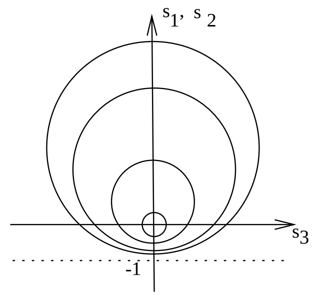

Figure 4: Deformed action of classical SU_2 on $\mathbb{R}{\rtimes}\mathbb{R}^2$

Of course one may take the n-fold tensor product of the Planck-scale quantum group, i.e. generators x_i, p_i and different i commuting. However, in higher dimensions the Ext_0 is much bigger and I do not know of any full computation of all the possibilities for $n > 1$. More interesting perhaps are some genuinely different higher-dimensional examples along similar but nonAbelian lines, one of which we describe now. The material is covered in [6], so we will be brief.

Thus, also from 1988, there is a bicrossproduct quantum group

$$\mathbb{C}(\mathbb{R}{\rtimes}\mathbb{R}^2){\blacktriangleright\!\!\triangleleft}U(su_2) \tag{26}$$

constructed in [28][33] (actually as a Hopf-von Neumann algebra; here we consider only the simpler algebraic structure underlying it.)

The nonAbelian group $\mathbb{R}{\rtimes}\mathbb{R}^2$ is the one whose enveloping algebra we have considered in Example 3 in Section 3 as noncommutative spacetime. Here, however, we take it with a Euclidean signature and a different notation. Explicitly, it consists of 3-vectors $\vec{s}$ with third component $s_3 > -1$ and with the 'curved $\mathbb{R}^3$' nonAbelian group law

$$\vec{s}\cdot\vec{t} = \vec{s} + (s_3+1)\vec{t}.$$

Its Lie algebra is spanned by x_0, x_i with relations $[x_i, x_0] = x_i$ for $i = 1, 2$ as discussed before (this is how this algebra appeared first, in [28], in connection this higher-dimensional version of the Planck-scale quantum group). Now, on the group $\mathbb{R}{\rtimes}\mathbb{R}^2$ there is an action of SU_2 by a deformed rotation. This is shown in Figure 4. The orbits are still spheres but non-concentrically nested and accumulating at $s_3 = -1$. This is a dynamical system and (26) is its Mackey quantisation as a cross product. We see that we have similar features as for the Planck-scale quantum group, including some kind of coordinate singularity as $s_3 = -1$.

At the same time there is a 'back reaction' of $\mathbb{R}{\rtimes}\mathbb{R}^2$ back on SU_2, which appears as a coaction of $\mathbb{C}(\mathbb{R}{\rtimes}\mathbb{R}^2)$ on $U(su_2)$ in the cross coalgebra structure of the quantum group. Therefore the dual system, related by Fourier theory or observable-state duality, is of the same form, namely

$$U(\mathbb{R}{\rtimes}\mathbb{R}^2){\triangleright\!\!\blacktriangleleft}\mathbb{C}(SU_2). \tag{27}$$

It consists of a particle on SU_2 moving under the action of $\mathbb{R}\rtimes\mathbb{R}^2$. This is the dual system which, in the present case, looks quite different.

Finally, the general theory of bicrossproducts allows for a 'Schroedinger representation' of (26) on $U(\mathbb{R}\rtimes\mathbb{R}^2)$ and similarly of its dual on $U(su_2)$. Such a picture means that the 'wave functions' live in these enveloping algebras viewed as noncommutative spaces. There are also more conventional Hilbert space representations as well.

4.3 General construction

There is a general construction for bicrossproduct quantum groups of which the ones discussed so far are all examples. Thus suppose that

$$X = GM$$

is a *factorisation of Lie groups*. Then one can show that G acts on the set of M and M acts back on the set of G such that X is recovered as a double cross product (simultaneously by the two acting on each other) $X \cong G \bowtie M$. This turns out to be just the data needed for the associated cross product and cross coproduct

$$\mathbb{C}(M) \blacktriangleright\!\!\triangleleft U(\mathfrak{g}) \tag{28}$$

to be a Hopf algebra. The roles of the two Lie groups is symmetric and the dual is

$$(\mathbb{C}(M) \blacktriangleright\!\!\triangleleft U(\mathfrak{g}))^* = U(\mathfrak{m}) \triangleright\!\!\blacktriangleleft \mathbb{C}(G) \tag{29}$$

which means that there are certain families of homogeneous spaces (the orbits of one group under the other) which come in pairs, with the algebra of observables of the quantisation of one being the algebra of expectation states of the quantisation of the other. This is the more or less purest form of the ideas of Section 2 based on Mach's principle[16] and duality.

On the other hand, factorisations abound in Nature. For example every complexification of a simple Lie group factorises into its compact real form G and a certain solvable group $G^\star$, i.e. $G_{\mathbb{C}} = GG^\star$. The notation here is of a modern approach to the Iwasawa decomposition in [28]. For example, $SL_2(\mathbb{C}) = SU_2 SU_2^\star$, where $SU_2^\star = \mathbb{R}\rtimes\mathbb{R}^2$, gives the bicrossproduct quantum group (26) in the preceding section. There are similar examples

$$\mathbb{C}(G^\star) \blacktriangleright\!\!\triangleleft U(\mathfrak{g}) \tag{30}$$

for all complex simple $\mathfrak{g}$. Also, slightly more general than the Iwasawa decomposition but still only a very special case of a general Lie group factorisation, let G be a Poisson-Lie group (a Lie group with a compatible Poisson-bracket). At the infinitesimal level the Poisson bracket defines a map $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ making $\mathfrak{g}$ into a Lie bialgebra. This is an infinitesimal idea of a quantum group and is such that $\mathfrak{g}^\star$ is also a Lie algebra. In this setting there is a Drinfeld double Lie bialgebra $D(\mathfrak{g})$ and its Lie group is an example of a factorisation $GG^\star$.

By the way, this is exactly the setting for nonAbelian Poisson-Lie T-duality [12] in string theory, for classical σ-models on G and $G^\star$. The quantum groups (30) and their duals are presumably related to the quantisations of the point-particle limit of these sigma models. If so this would truly extend T-duality to the quantum case via the above observable-state duality ideas. While this is not proven exactly, something *like* this appears to be the case. Moreover, the bicrossproduct duality for (28) is much more general and is not limited to such Poisson-Lie structures on G. The group M need not be dual to G in the above sense and need not even have the same dimension. Recently it was shown that the Poisson-Lie T-duality in a Hamiltonian (but not Lagrangian) setting indeed generalises to a general factorisation like this[13].

Finally, there is one known connection between the bicrossproduct quantum groups and the more standard $U_q(\mathfrak{g})$ which we will consider next. Namely, Lukierski et al.[30] showed that a certain contraction process turned $U_q(so_{3,2})$ in a certain limit to some kind of 'κ-deformed' Poincaré algebra as mentioned below Example 3 in Section 3. It turned out later[29] that this was isomorphic to one of the bicrossproduct Hopf algebras above,

$$_\kappa\mathrm{Poincare}\cong\mathbb{C}(\mathbb{R}\rtimes\mathbb{R}^3)\blacktriangleright\!\!\triangleleft U(so_{3,1}).$$

The isomorphism here is nontrivial (which means in particular that κ-Poincaré certainly arose independently of the early bicrossproducts such as the 3-dimensional case (26)). On the other hand, the bicrossproduct version of κ-Poincaré from [29] brought many benefits. First of all, the Lorentz sector is *undeformed.* Secondly, the dual is easy to compute (being an example of the general self-duality ideas above) and, finally, the Schroedinger representation means that this quantum group indeed acts covariantly on $U(\mathbb{R}\rtimes\mathbb{R}^3)$, which should therefore be viewed as the κ-Minkowski space appropriate to this κ-Poincaré (prior to [29] one had only the noncovariant action of it on usual commutative Minkowski space, leading to a number of inconsistencies in attempting to model physics based on κ-Poincaré alone). Of course the point of view of Poincaré algebra as symmetry appears at first different from the main point of view of bicrossproducts as the quantisations of a dynamical system. However, as in Section 2 (and even for the classical Poincaré algebra) a symmetry enveloping algebra *should* also appear as part of (or all of) the quantum algebra of observables of the associated quantum theory because it should be realised among the quantum fields[14].

5 Deformed quantum enveloping algebras

No introduction to quantum groups would be complete if we did not also mention the much more well known deformations $U_q(\mathfrak{g})$ of complex simple $\mathfrak{g}$ arising from inverse scattering and the theory of solvable lattice models[7][8]. These have not, however, been very directly connected with Planck scale physics (although there are some recent proposals for this, as we saw in the lectures of Lee Smolin). They certainly did not arise that way and are not the quantum algebras

of observables of physical systems. Therefore this is only going to be a lightning introduction to this topic. For more, see [6][10][34].

Rather, these quantum groups $U_q(\mathfrak{g})$ arise naturally as 'generalised' symmetries of certain spin chains and as generalised symmetries in the Wess-Zumino-Witten model conformal field theory. Just as groups can be found as symmetries of many different and unconnected systems, the same is true for the quantum groups $U_q(\mathfrak{g})$. They do, however, have a perhaps richer and more complex mathematical structure than the bicrossproducts, which is what we shall briefly outline.

As Hopf algebras one has the same duality ideas nevertheless. Thus, the quantum group $U_q(su_2)$ with generators $H, X_\pm$ and relations and coproduct

$$[H, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}$$

$$\Delta X_\pm = X_\pm \otimes q^{\frac{H}{2}} + q^{\frac{-H}{2}} \otimes X_\pm, \quad \Delta H = H \otimes 1 + 1 \otimes H$$

is dual to the quantum group $\mathbb{C}_q(SU_2)$ generated by a matrix of generators a, b, c, d. This has six relations of q-commutativity

$$ba = qab,\ ca = qac,\ bc = cb,\ dc = qcd,\ db = qbd,\ da = ad + (q - q^{-1})bc$$

and a determinant relation $ad - q^{-1}bc = 1$. The pairing is the same as in Example 2 in Section 2 at the level of generators (after a change of basis).

The main feature of these quantum groups, in contrast to the bicrossproduct ones, is that their representations form braided categories. Thus, if $V, W \in \mathrm{Rep}(U_q(\mathfrak{g}))$ then $V \otimes W$ is (as for any quantum group) also a representation. The action is

$$h \triangleright (v \otimes w) = (\Delta h).(v \otimes w) \tag{31}$$

for all $h \in U_q(\mathfrak{g})$, where we use the coproduct (for example the linear form of the coproduct of H means that it acts additively). The special feature of quantum groups like $U_q(\mathfrak{g})$ is that there is an element $\mathcal{R} \in U_q(\mathfrak{g}) \bar{\otimes} U_q(\mathfrak{g})$ (the 'universal R-matrix or quasitriangular structure') which ensures an isomorphism of representations by

$$\Psi_{V,W} : V \otimes W \to W \otimes V, \quad \Psi_{V,W}(v \otimes w) = P \circ \mathcal{R}.(v \otimes w) \tag{32}$$

where P is the usual permutation or flip map. This *braiding* Ψ behaves much like the usual transposition or flip map for vector spaces but does not square to one. To reflect this one writes $\Psi =$ ⤬, $\Psi^{-1} =$ ⤫. It has properties consistent with the braid relations, i.e. when two braids coincide the compositions of Ψ, Ψ^{-1} that they represent also coincide. The fundamental braid relation of the braid group in Figure 5(a) corresponds to the famous Yang-Baxter or braid relation for the matrix corresponding to Ψ.

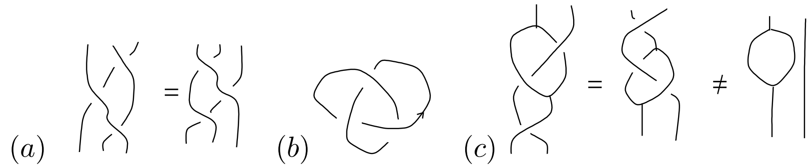

Figure 5: (a) Braid relations (b) Trefoil knot (c) Braided algebra calculation

From this it is more or less obvious that such quantum groups lead to knot invariants. One can scan the (oriented) knot such as in Figure 5(b) from top to bottom. We choose a representation V with dual V^* and label the knot by V against a downward arc and V^* against an upward arc. As we read the knot, when we encounter an arc ${}_V\cap_{V^*}$ we let it represent the canonical element $\sum_a e_a \otimes f^a \in V \otimes V^*$. When we encounter crossings we represent them by the appropriate Ψ and finally when we encounter ${}^{V^*}\cup^{V}$ we apply the evaluation map. There is also a prescription for when we encounter ${}_{V^*}\cap_V$ and ${}^V\cup^{V^*}$. At the end of the day we obtain a number depending on q (which went into the braiding). This function of q is (with some fiddling that we have not discussed) an invariant of the knot regarded as a framed knot. This is not the place to give details of knot theory, but this is the rough idea. In physical terms one should think of the knot as a process in 1+1 dimensions in which a particle V and antiparticle V^* is created at an arc, some kind of scattering Ψ occurs at crossings, etc.

For standard $U_q(\mathfrak{g})$ the construction of representations is not hard, all the standard ones of $\mathfrak{g}$ just q-deform. For example, the spin $\frac{1}{2}$ representation of su_2 deforms to a 2-dimensional representation of $U_q(su_2)$. The associated knot invariant is the celebrated Jones polynomial.

5.1 Braided mathematics and braided groups

This braiding is the key property of the quantum groups $U_q(\mathfrak{g})$ and other 'quasitriangular Hopf algebras' of similar type. It means in particular that *any algebra on which the quantum group acts covariantly becomes braided*. This is therefore indicative of a whole braided approach to noncommutative geometry or *braided geometry* via algebras or 'braided' spaces on which quantum groups $U_q(\mathfrak{g})$ act as generalised symmetries. Note that we are not so much interested in this point of view in the noncommutative geometry of the quantum groups $U_q(\mathfrak{g})$ themselves, although one can study this as a source of mathematical examples. More physical is the algebras in which these objects act.

In this approach the meaning of q is that it enters into the braiding, i.e. it generalises the -1 of supertransposition in super-geometry. This is 'orthogonal' to the usual idea of noncommutative geometry, i.e. it is not so much a property of one algebra but of composite systems, namely of the noncommutativity of tensor products. The simplest new case is where the braiding is just a factor q. To see how this works, consider the *braided line* $B = \mathbb{C}[x]$. As an algebra this is just the polynomials in one variable again.

Example 4 *Let $B = \mathbb{C}[x]$ be the braided line, where independent copies x, y have braid statistics $yx = qxy$ when one is transposed past the other (c.f. a Grassmann variable but with -1 replaced by q). Then*

$$\partial_q f(y) = x^{-1}\left(f(x+y) - f(x)\right)|_{x=0} = \frac{f(y) - f(qy)}{(1-q)y}$$

This is easy to see on monomials, i.e. $\partial_q y^n$ is the coefficient of the x-linear part in $(x+y)^n$ after we move all x to the left. In fact mathematicians have played with such a q-derivative since 1908[35] as having many cute properties. We see[36] that it arises very naturally from the braided point of view – one just has to realise that x is a braided variable. This point of view also leads to the correct properties of integration. Namely there is a relevant indefinite integration to go with ∂_q characterised by[37]

$$\int_0^{x+y} f(z)\mathrm{d}_q z = \int_0^y f(z)\mathrm{d}_q z + \int_0^x f(z+y)\mathrm{d}_q z \tag{33}$$

provided $yx = qxy$, $yz = qzy$ etc., during the computation. In the limit this gives the infinite Jackson integral previously known in this context. One also has braided exponentials, braided Fourier theory etc., for these braided variables.

The braided point of view is also much more powerful than simply trying to sprinkle q into formulae here and there.

Example 5 *Let $B = \mathbb{C}_q^2$ be the quantum-braided plane generated by x, y with the relations $yx = qxy$, where two independent copies have the braid statistics*

$$x'x = q^2 xx', \quad x'y = qyx', \quad y'y = q^2 yy', \quad y'x = qxy' + (q^2-1)yx'.$$

Here x', y' are the generators of the second copy of the plane. Then

$$(y+y')(x+x') = q(x+x')(y+y')$$

i.e. $x+x', y+y'$ is another copy of the quantum-braided plane. Then by similar definitions as above, one has braided partial derivatives

$$\partial_{q,x} f(x,y) = \frac{f(x,y) - f(qx,y)}{(1-q)x}, \quad \partial_{q,y} f(x,y) = \frac{f(qx,y) - f(qx,qy)}{(1-q)y}$$

for expressions normal ordered to x on the left. Note in the second expression an extra q as $\partial_{q,y}$ moves past the x

Thus *you can add points in the braided plane*, and then (by an infinitesimal addition) define partial derivatives etc. This is a problem (multilinear q-analysis) which had been open since 1908 and was only solved relatively recently (by the author) in [36], as a demonstration of braided mathematics. We note in passing that $yx = qxy$ is sometimes called the 'Manin plane'. Manin

considered only the algebra and a quantum group action on it, without the braided point of view, without the braided addition law and without the partial derivatives.

Finally, there is a more formal way by which all such constructions are done systematically, which we now explain. It amounts to nothing less than a new kind of algebra in which algebraic symbols are replaced by braids and knots.

First of all, given two algebras B, C in a braided category (such as the representation of $U_q(\mathfrak{g})$) we have a braided tensor product $B\underline{\otimes}C$ algebra in the same category defined like a superalgebra but with -1 replaced by the braiding $\Psi_{C,B}$. Thus the tensor product becomes noncommutative (even if each algebra B, C was commutative) – the two subalgebras 'commute' up to Ψ. This is the mathematical definition of *braid statistics: the noncommutavity of the notion of 'independent' systems*. We call such noncommutativity *outer* in contrast to the inner noncommutativity of quantisation, which is a property of one algebra alone. In Example 4, the joint algebra of the independent x, y is $\mathbb{C}[x]\underline{\otimes}\mathbb{C}[y]$ with $\Psi(x \otimes y) = qy \otimes x$. In Example 5 the braided tensor product is between one copy x, y and the other x', y'. The braiding Ψ in this case is more complicated. In fact it is the same braiding from the $U_q(su_2)$ spin $\frac{1}{2}$ representation that gave the Jones polynomial. The miracle that makes knot invariants is the same miracle that allows braided multilinear algebra.

The addition law in both the above examples makes them into braided groups[38]. They are like quantum groups or super-quantum groups but with braid statistics. Thus, there is a coproduct

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta y = y \otimes 1 + 1 \otimes y$$

etc., (this is a more formal way to write $x + x', y + y'$). But $\Delta : B \to B\underline{\otimes}B$ rather than mapping to the usual tensor product. We do not want to go into the whole theory of braided groups here. Suffice it to say that the theory can be developed to the same level as quantum groups: integrals, Fourier theory, etc., but using new techniques. One draws the product $B \otimes B \to B$ as a map $\curlyvee$, the coproduct as $\curlywedge$, etc. Similarly with other maps, some strands coming in for the inputs and some leaving for the outputs. We then 'wire up' an algebraic expression by wiring outputs of one operation into the inputs of others. When wires have to cross under or over we have to chose one or the other as Ψ or Ψ^{-1}. We draw such diagrams flowing down the page. An example of a braided-algebra calculation is given in Figure 5(c).

Braided groups exist in abundance. There are general arguments that every algebraic quantum field theory contains at its heart some kind of (slightly generalised) braided group[39]. Moreover, the ideas here are clearly very general: braided algebra.

5.2 Systematic q-Special Relativity

Clearly braided groups are the correct foundation for q-deformed geometry based on q-planes and similar q-spaces. One of their main successes in the period 1992-1994 was a more or

less complete and systematic q-deformation by the team in Cambridge of the main structures of special relativity and electromagnetism, i.e. q-Minkowski space and basic structures [40][41][36][42][43][37][44][45][46]:

- q-Minkowski space as 2×2 braided Hermitian matrices
- q-addition etc., on q-Minkowski space
- q-Lorentz quantum group $\mathbb{C}_q(SU_2)\bowtie\mathbb{C}_q(SU_2)$
- q-Poincaré+scale quantum group $\mathbb{R}_q^{1,3}{>\!\!\!\triangleleft}\widetilde{U_q(so_{1,3})}$
- q-partial derivatives
- q-differential forms
- q-epsilon tensor
- q-metric
- q-integration with Gaussian weight
- q-Fourier theory
- q-Green functions (but no closed form)
- q-$*$ structures and q-Wick rotation

The general theory works for any braiding or 'R-matrix'. I do want to stress, however, that this project was not in a vacuum. For example, the algebra of q-Minkowski had been proposed independently of [40] in [47], but without the braided matrix or additive structures. The q-Lorentz was studied by the same authors but without its quasitriangular structure, Wess, Zumino et al.[48] studied the q-Poincaré but without its semidirect structure and action on q-Minkowski space, while Fiore[49] studied q-Gaussians in the Euclidean case, etc. More recently, we have[50][51],

- q-conformal group $\mathbb{R}_q^{1,3}{>\!\!\!\triangleleft}\widetilde{U_q(so_{1,3})}{\triangleright\!\!\!<}\mathbb{R}_q^{1,3}$
- q-diffeomorphism group

Notably not on the list, in my opinion still open, is the correct formulation of the q-Dirac equation. Aside from this, the programme came to an end when certain deep problems emerged. In my opinion they are as follows. First of all, we ended up with formal power-series e.g. the q-Green function is the inverse Fourier transform of $(\vec{p}\cdot\vec{p}-m^2)^{-1}$ so in principle it is now defined. But not in closed form! The methods of q-analysis as in [35][52] are not yet far enough advanced to have nice names and properties for the kinds of powerseries functions encountered.

This is a matter of time. Similarly, braided integration means we can in principle write down and compute braided Feynman diagrams and hence define braided quantum field theory at least operatively. Recently R. Oeckl was able[53] to apply the braided integration theory of [37] not to q-spacetime but directly to a q-coordinate algebra as the underlying vector space of fields on spacetime. Here the braided algebra B replaces the 'fields' on spacetime. Choosing a basis of such fields one can still apply braided Gaussian integration and actually compute correlation functions. So the computational problems can and are being overcome.

Secondly and more conceptual, it should be clear that when we deform classical constructions to braided ones we have to choose Ψ or Ψ^{-1} whenever wires cross. Sometimes neither will do, things get tangled up. But if we succeed it means that for every q-deformation there is another where we could have made the opposite choice in every case. *This classical geometry bifurcates into two q-deformed geometries according to* Ψ *or* Ψ^{-1}. Moreover, the role of the $*$ operation is that it interchanges these two[45]. Roughly speaking,

$$\begin{array}{ccc} & \nearrow & q-\text{geometry} \\ \text{classical geometry} & & \updownarrow * \\ & \searrow & \text{conjugate } q-\text{geometry} \end{array}$$

where the conjugate is constructed by interchanging the braiding with the inverse braiding (i.e. reversing braid crossings in the diagrammatic construction). For the simplest cases like the braided line it means interchanging q, q^{-1}. This is rather interesting given that the $*$ is a central foundation of quantum mechanics and our concepts of probability. But it also means one cannot do q-quantum mechanics etc., with q-geometry alone; one needs also the conjugate geometry.

5.3 The physical meaning of q

According to what we have said above, the true meaning of q is that it generalises the -1 of fermionic statistics. That is why it is dimensionless. It is nothing other than a parameter in a mathematical structure (the braiding) in a generalisation of our usual concepts of algebra and geometry, going a step beyond supergeometry.

This also means that q is an ideal parameter for regularising quantum field theory. Since most constructions in physics q-deform, such a regularisation scheme is much less brutal than say dimensional or Pauli-Villars regularisation as it preserves symmetries as q-symmetries, the q-epsilon tensor etc. [15]. In this context it seems at first *too good* a regularisation. Something has to go wrong for anomalies to appear.

Conjecture 4 *In q-regularisation the fact that only the Poincaré+scale q-deforms (the two get mixed up) typically results (when the regulator is removed after renormalisation) in a scale anomaly of some kind.*

This is probably linked to a much nicer treatment of the renormalisation group that should be possible in this context. Again a lot of this must await more development of the tools of

q-analysis. At any rate the result in [15] is that q-deformation does indeed regularise, turning some of the infinities from a Feynman loop integration into poles $(q-1)^{-1}$.

All of this is related to the Planck scale as follows. Thus, as well as being a good regulator one can envisage (in view of our general ideas about noncommutativity and the Planck scale) that the actual world is in reality better described by $q \neq 1$ due to Planck scale effects. In other words q-deformed geometry could indeed be the next-to-classical order approximation to the geometry coming out of some unknown theory of quantum gravity. This was the authors own personal reason[15] for spending some years q-deforming the basic structures of physics. The UV cut-off provided by a 'foam-like structure of space time' would instead be provided by q-regularisation with $q \neq 1$. Moreover, if this is so then q-deformed quantum field theory should also appear coming out of quantum gravity as an approximation one better than the usual. Such a theory would be massless according to the above remarks (because there is no q-Poincaré without the scale generator). Or at least particle masses would be small compared to the Planck mass. *How the q-scale invariance breaks would then be a mechanism for mass generation.*

There are also several other 'purely quantum' features of q-geometry not visible in classical geometry, which would likewise have consequences for Planck scale physics. One of them is:

Theorem 2 *The braided group version of the enveloping algebras $U_q(\mathfrak{g})$ and their q-coordinate algebras are isomorphic. I.e. there is essentially only one object in q-geometry with different scaling limits as $q \to 1$ to give the classical enveloping algebra of $\mathfrak{g}$ or coordinate algebra of G.*

The self-duality isomorphisms involve dividing by $q-1$ and are therefore singular when $q = 1$, i.e. this is totally alien to conventional geometric ideas. Enveloping algebras and their coordinate algebras are supposed to be dual not isomorphic. This self-duality in q-geometry is rather surprising but is fully consistent with the self-duality ideas of Section 2. In many ways q-geometry is simpler and more regular than the peculiar $q = 1$ that we are more familiar with.

Recently, it was argued[11] that since loop gravity is linked to the Wess-Zumino-Witten model, which is linked to $U_q(su_2)$ (or some other quantum group), that indeed q-geometry should appear coming out of quantum-gravity with cosomological constant Λ. There is even provided a formula

$$q = e^{\frac{2\pi\imath}{2+k}}, \quad k = \frac{6\pi}{G_{\rm Newton}^2\Lambda}.$$

If so then the many tools of q-deformation developed in the last several years would suddenly be applicable to study the next-to-classical structure of quantum-gravity. The fact that loop variable and spin-network methods 'tap into' the revolutions that have taken place in the last decade around quantum groups, knot theory and the WZW model (this was evident for example in the black-hole entropy computation[54]) makes such a conjecture reasonable. It also indicates to me that these new quantum gravity methods are not just 'pushing some problem off to another corner' but are building on a certain genuine advance that has already revolutionised several

other branches of mathematics. Usually in science when one big door is opened it has nontrivial repercussions in several fields.

One way or another the general idea is that quantum effects dominant at the Planck scale force geometry itself to be modified as we approach it such as to have a noncommutative or 'quantum' aspect expressed by $q \neq 1$. Although q is dimensionless and might be given, for example, by formulae such as the above, one can and should still think of q as behaving formally like the exponential of an *effective Planck's constant* $\hbar_0$, say. That is we can make semiclassical expansions, speak of Poisson-brackets being 'quantized' etc. This is not exactly physical quantisation except in so far as quantum effects at the Planck scale are at the root of it. The precise physical link can only be made in a full theory of quantum gravity. It is only in this sense, however, that q-geometry is 'quantum geometry' and 'quantum groups' are so called. For example, the q-coordinate algebras of $U_q(\mathfrak{g})$ are quantisations in this sense of a certain Poisson-Lie bracket on G (as mentioned in Section 4.3). Similarly for all our other q-spaces.

Example 6 *[50] q-Minkowski space quantises a Poisson-bracket on* $\mathbb{R}^{1,3}$ *given by the action of the special conformal translations.*

This again points to a remarkable interplay between q-regularisation, the renormalisation group, gravity and particle mass.

At least in this context we want to note that the braided approach of this subsection gives a new and systematic approach to the 'quantisation' problem that solves by new 'braid diagram' methods some age-old problems. Usually, one writes a Poisson bracket and tries to 'quantise' it by a noncommutative algebra. Apart from existence, *the problem often overlooked is what I call the uniformity of quantisation problem.* There is only one universe. How do we know when we have quantised this or that space separately that they are consistent with each other, i.e. that they all fit together to a single quantum universe?

Our theme in Section 2 is of course is that quantisation is not a well-defined problem. Rather one *should* have a deeper point of view which leads directly to the quantum-algebraic world – what we call geometry is then the semiclassical limit of the intrinsic structure of that, i.e. all different spaces and choices of Poisson structures on them will emerge from semiclassicalisation and not vice versa.

Braided algebra solves the uniformity problem in this way. Apart from giving the q-deformation of most structures in physics, it does it uniformly and in a generally consistent way because what what we deform is actually the category of vector spaces into a braided category. All constructions based on linear maps then deform coherently and consistently with each other as braid diagram constructions (so long as they do not get tangled). After that one inserts the formulae for specific braidings (e.g. generated by specific quantum groups) to get the q-deformation formulae. *After that* one semiclassicalises by taking commutators to lowest order, to get the Poisson-bracket that we have just quantised. Moreover, different quantum groups

$U_q(\mathfrak{g})$ are all mutually consistent being related to each other by an inductive construction[55]. We have seen this with q-Minkowski space above.

In summary, the q-deformed examples demonstrate a remarkable unification of three different points of view; q as a generalisation of fermionic -1, q as a 'quantisation' (so these ideas are unified) and q as a powerful regularisation parameter in physics. By the way, these are all far from the original physical role of q, where $U_q(su_2)$ arose as a generalised symmetry of the XXY lattice model and where q measures the anisotropy due to an applied external magnetic field (rather, they are the authors' point of view developed under the heading of the braided approach to q-deformation and braided geometry).

6 Noncommutative differential geometry and Riemannian manifolds

We have promised that today there is a more or less complete theory of noncommutative differential geometry that includes most of the naturally occurring examples such as those in previous sections, but is a general theory not limited to special examples and models, i.e. has the same degree of 'flabbiness' as conventional geometry. Here I will try to convince you of this and give a working definition of a 'quantum manifold' and 'quantum Riemannian manifold'[4]. I do not want to say that this is the last word; the subject is still evolving but there is now something on the table. Among other things, our constructions are purely algebraic with operator and C^*-algebra considerations as in Connes' approach not fully worked. In any case, the reader may well want to start with the more accessible Section 6.4, where we explore the semiclassical implications at the more familiar level of the ordinary differential geometry coming out of the full noncommutative theory.

6.1 Quantum differential forms

As explained in Section 2 our task is nothing other than to give a formulation of geometry where the coordinate algebra on a manifold is replaced by a general algebra M. The first step is to *choose* the cotangent space or differential structure. Since one can multiply forms by 'functions' from the left and right, the natural definition is to define a first order calculus as a bimodule Ω^1 of the algebra M, along with a linear map $\mathrm{d} : M \to \Omega^1$ such that

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b, \quad \forall a, b \in M.$$

Differential structures are not unique even classically, and even more non-unique in the quantum case. There is, however, one universal example of which others are quotients. Here

$$\Omega^1_{\mathrm{univ}} = \ker \cdot \subset M \otimes M, \quad \mathrm{d}a = a \otimes 1 - 1 \otimes a.$$

Classically we do not think about this much because on a group there *is* a unique translation-invariant differential calculus; since we generally work with manifolds built on or closely related

to groups we tend to take the inherited differential structure without thinking. In the quantum case, i.e. when M is a quantum (or braided) group one has a similar notion [56]: a differential calculus is bicovariant if there are coactions $\Omega^1 \to \Omega^1 \otimes M, \Omega^1 \to M \otimes \Omega^1$ forming a bicomodule and compatible with the bimodule structures and d.

Theorem 3 *[57] For the q-coordinate rings of the quantum groups $U_q(\mathfrak{g})$, the (co)irreducible bicovariant (ω^1, d) are essentially (for generic q) in correspondence with the irreducible representations ρ of $\mathfrak{g}$, and*

$$\Omega^1_{\rm univ} = \oplus_\rho \Omega^1_\rho.$$

The lowest spin $\frac{1}{2}$ representation of $U_q(su_2)$ defines its usual differential calculus plus a Casimir as $q \to 1$. The higher differential calculi show up in the q-geometry and correspond to higher spin. This should therefore be a step towards understanding how macroscopic differential geometry arises out of the loop gravity and spin network formalism. For example, the black-hole entropy computation[54] reported in Abbay Ashtekar's lectures at the conference was dominated by the spin $\frac{1}{2}$ states, which seems to me should be analogous to the standard differential calculus on the spin connection bundle dominating as macroscopic geometry emerges from the quantum gravity theory.

We do not have room to give more details here even of an example of Theorem 3, but see [57]. Instead we content ourselves with an even simpler and more pedagogical result.

Proposition 1 *[58] If k is a field and $M = k[x]$ the polynomials in one variable, the (co)irreducible bicovariant calculi (Ω^1, d) are in correspondence with field extensions of the form $k_\lambda = k[\lambda]$ modulo $m(\lambda) = 0$, where m is an irreducible monic polynomial. Here*

$$\Omega^1 = k_\lambda[x], \quad \mathrm{d}f(x) = \frac{f(x+\lambda) - f(x)}{\lambda},$$

$$f(x).g(\lambda, x) = f(x+\lambda)g(\lambda, x), \quad g(\lambda, x).f(x) = g(\lambda, x)f(x)$$

for functions f and one-forms g.

For example, over $\mathbb{C}$, (Ω^1, d) on $\mathbb{C}[x]$ are classified by $\lambda_0 \in C$ and one has

$$\Omega^1 = \mathrm{d}x\mathbb{C}[x], \quad \mathrm{d}f = \mathrm{d}x\frac{f(x+\lambda_0) - f(x)}{\lambda_0}, \quad x\mathrm{d}x = (\mathrm{d}x)x + \lambda_0. \tag{34}$$

We see that the Newtonian case $\lambda_0 = 0$ is only one special point in the moduli space of quantum differential calculi. But if Newton had not supposed that differentials and forms commute he would have had no need to take this limit. What one finds with noncommutative geometry is that there is no need to take this limit at all. In particular, noncommutative geometry extends our usual concepts of geometry to lattice theory without taking the limit of the lattice spacing going to zero.

It is also interesting that the most important field extension in physics, $\mathbb{R} \subset \mathbb{C}$, can be viewed noncommutative-geometrically with complex functions $\mathbb{C}[x]$ the quantum 1-forms on the algebra of real functions $\mathbb{R}[x]$. As such its quantum cohomology is nontrivial, see [58].

6.2 Bundles and connections

To go further one has to have a pretty abstract view of differential geometry. For trivial bundles it is a little easier: fix a quantum group coordinate ring H. Then a gauge field is a map $H \to \Omega^1$, etc. See [59][60]. To define a manifold, however, one has to handle nontrivial bundles. In noncommutative geometry there is (as yet) no proper way to build this by patching trivial bundles. All those usual concepts involve open sets etc, not existing in the noncommutative case. Fortunately, if one thinks about it abstractly enough one can come up with a purely algebraic formulation independent of any patches or coordinate system. For simplicity we are going to limit attention to the universal calculi; the theory is know for general calculi as well.

Basically, a classical bundle has a free action of a group and a local triviality property. In our algebraic terms this translates[5][60] to an algebra P in the role of 'coordinate algebra of the total space of the bundle', a coaction $\Delta_R : P \to P \otimes H$ of the quantum group H such that the fixed subalgebra is M,

$$M = P^H = \{p \in P |\ \Delta_R p = p \otimes 1\}. \tag{35}$$

Local triviality is replaced by the requirement that

$$0 \to P(\Omega^1 M)P \to \Omega^1 P \xrightarrow{\tilde{\chi}} P \otimes \ker \epsilon \to 0 \tag{36}$$

is exact, where $\tilde{\chi} = (\cdot \otimes \mathrm{id})\Delta_R$ plays the role of generator of the vertical vector fields corresponding classically to the action of the group (for each element of H^* it maps $\Omega^1 P \to P$ like a vector field). Exactness says that the one-forms $P(\Omega^1 M)P$ lifted from the base are exactly the ones annihilated by the vertical vector fields.

An example is the quantum sphere. Classically the inclusion $U(1) \subset SU_2$ in the diagonal has coset space S^2 and defines the $U(1)$ bundle over the sphere on which the monopole lives. The same idea works here, but since we deal with coordinate algebras the arrows are reversed. The coordinate algebra of $U(1)$ is the polynomials $\mathbb{C}[g, g^{-1}]$.

Example 7 *There is a projection from $\mathbb{C}_q(SU_2) \to \mathbb{C}[g, g^{-1}]$*

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

Its induced coaction $\Delta_R = (\mathrm{id} \otimes \pi)\Delta$ is by the degree defined as the number of a, c minus the number of b, d in an expression. The quantum sphere S^2_q is the fixed subalgebra i.e. the degree zero part. Explicitly, it is generated by $b_3 = ad$, $b_+ = cd$, $b_- = ab$ with q-commutativity relations

$$b_\pm b_3 = q^{\pm 2} b_3 b_\pm + (1 - q^{\pm 2}) b_\pm, \quad q^2 b_- b_+ = q^{-2} b_+ b_- + (q - q^{-1})(b_3 - 1)$$

and the sphere equation $b_3^3 = b_3 + q b_- b_+$, and forms a quantum bundle[5][60].

When $q \to 1$ we can write $b_\pm = \pm(x \pm \imath y)$, $b_3 = z + \frac{1}{2}$ and the sphere equation becomes $x^2 + y^2 + z^2 = \frac{1}{4}$ while the others become that x, y, z commute. The quantum sphere itself is a member of a 2-parameter family[61] of quantum spheres (the others can also be viewed as bundles in a suitable framework[62].)

One can go on and define a connection as an equivariant splitting

$$\Omega^1 P = P(\Omega^1 M)P \oplus \text{complement} \tag{37}$$

i.e. an equivariant projection Π on $\Omega^1 P$. One can show the required analogue of the usual theory, i.e. that such a projection corresponds to a connection form such that

$$\omega : \ker \epsilon \to \Omega^1 P, \quad \tilde{\chi}\omega = \mathrm{id} \tag{38}$$

where ω intertwines with the adjoint coaction of H on itself. There is such a connection on the example above – the q-monopole[5]. It is $\omega(g-1) = d\mathrm{d}a - qb\mathrm{d}c$.

Finally, one can define associated bundles. If V is a vector space on which H coacts then we define the associated 'bundles' $E^* = (P \otimes V)^H$ and $E = \hom_H(V, P)$, the space of intertwiners. The two bundles should be viewed geometrically as 'sections' in classical geometry of bundles associated to V and V^*. Given a suitable (strong) connection one has a covariant derivative

$$D_\omega : E \to E \otimes M, \quad D_\omega = (\mathrm{id} - \Pi) \circ \mathrm{d} \tag{39}$$

This is where the noncommutative differential geometry coming out of quantum groups links up with the more traditional C^*-algebra approach of A. Connes and others. Traditionally a vector bundle over any algebra is defined as a finitely generated projective module. However, there is no notion of quantum principal bundle of course without quantum groups. The associated bundles to the q-monopole bundle are indeed finitely generated projective modules[63]. The projectors are elements of the noncommutative K-theory $K_0(S^2_q)$ and their pairing with Connes' cyclic cohomology[9] allows one to show that the bundle is non-trivial even when $q \neq 1$. Thus the quantum groups approach is compatible with Connes' approach but provides more of the (so far algebraic) infrastructure of differential geometry – principal bundles, connection forms, etc. otherwise missing.

6.3 Soldering and quantum Riemannian structure

With the above ingredients we can give a working definition of a quantum manifold. See refer to [4] for details. The idea is that the main feature of being a manifold is that, locally, one can chose a basis of the tangent space at each point (e.g. a vierbein in physics) patching up globally via GL_n gauge transformations. In abstract terms it means a frame bundle to which the tangent bundle is associated by a 'soldering form'. For a general algebra M we specify this 'frame bundle' directly as some suitable quantum group principal bundle.

Thus, we define a *frame resolution* of M as quantum principal bundle (P, H, Δ_R) over M, a comodule V and an equivariant 'soldering form' $\theta : V \to P\Omega^1 M \subset \Omega^1 P$ such that the induced map

$$E^* \to \Omega^1 M, \quad p \otimes v \mapsto p\theta(v) \tag{40}$$

is an isomorphism. Of course, all of this has to be done with suitable choices of differential calculi on M, P, H whereas we have been focusing for simplicity on the universal calculi. There are some technical problems here but the same definitions more or less work in general. Our working definition[4] of a *quantum manifold* is this data $(M, \Omega^1, P, H, \Delta_R, V, \theta)$.

The definition works in that one has many usual results. For example, a connection ω on the frame bundle induces a covariant derivative D_ω on the associated bundle E which maps over under the soldering isomorphism to a covariant derivative

$$\nabla : \Omega^1 M \to \Omega^1 M \underset{M}{\otimes} \Omega^1 M. \tag{41}$$

Its torsion is defined as corresponding similarly to $D_\omega \theta$.

Defining a Riemannian structure is harder. It turns out that it can be done in a 'self-dual' manner as follows. Given a framing, a 'generalised metric' isomorphism $\Omega^{-1} M \to \Omega^1 M$ between vector fields and one forms can be viewed as the existence of *another* framing $\theta^* : V^* \to (\Omega^1 M)P$, which we call the *coframing*, this time with V^*. Nondegeneracy of the metric corresponds to θ^* inducing an isomorphism $E \cong \Omega^1 M$.

Thus our working definition[4] of a quantum Riemannian manifold is the data $(M, \Omega^1, P, H, \Delta_R, V, \theta, \theta^*)$, where we have a framing and at the same time $(M, \Omega^1, P, H, \Delta_R, V^*, \theta^*)$ is another framing. The associated quantum metric is

$$g = \sum_a \theta^*(f^a)\theta(e_a) \in \Omega^1 M \underset{M}{\otimes} \Omega^1 M \tag{42}$$

where $\{e_a\}$ is a basis of V and $\{f^a\}$ is a dual basis (c.f. our friend the canonical element exp from Fourier theory in Section 3).

Now, this self-dual formulation of 'metric' as framing and coframing is symmetric between the two. One could regard the coframing as the framing and vice versa. From our original point of view its torsion tensor corresponding to $D_\omega \theta^*$ is some other tensor, which we call the *cotorsion tensor*[4]. We then define a generalised Levi-Civita connection on a quantum Riemannian manifold as the ∇ of a connection ω such that the torsion and cotorsion tensors both vanish.

This is about as far as this programme has reached at present. One defines curvature of course as corresponding to the curvature of ω, which is $d\omega + \omega \wedge \omega$, but before we can finish the program outlined in Section 2 we still need to understand the Ricci and Einstein tensors in this setting. For this one has to understand their classical meaning more abstractly i.e. beyond some contraction formulae even in conventional geometry. It would appear that it has a lot to do with entropy and the relation between gravity and counting (geometric) states thermodynamically.

6.4 Semiclassical limit

To get the physical meaning of the cotorsion tensor and other ideas coming out of noncommutative Riemannian geometry, let us consider the semiclassical limit. *What we find is that noncommutative geometry forces us to slightly generalise conventional Riemannian geometry itself.* If noncommutative geometry is closer to what comes out of quantum gravity then this generalisation of conventional Riemannian geometry should be needed to include Planck scale effects or at least to be consistent with them when they emerge at the next order of approximation.

The generalisation, more or less forced by the noncommutativity, is as follows:

- We have to allow any group G in the 'frame bundle', hence the more general concept of a 'frame resolution' (P, G, V, θ^a_μ) or *generalised manifold*.
- The *generalised metric* $g_{\mu\nu} = \sum_a \theta^{*a}_\mu \theta_{\nu a}$ corresponding to a coframing θ^{*a}_μ is nondegenerate but need not be symmetric.
- The *generalised Levi-Civita* connection defined as having vanishing torsion and vanishing cotorsion respects the metric only in a skew sense

$$\nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\mu\rho} = 0 \tag{43}$$

- The group G is not unique (different flavours of frames are possible, e.g. an E_6-resolved manifold), not necessarily based on SO_n. This gives different flavours of covariant derivative ∇ that can be induced by a connection form ω.
- Even when G is fixed and $g_{\mu\nu}$ is fixed, the generalised Levi-Cevita condition does not fix ∇ uniquely, i.e. one should use a first order $(g_{\mu\nu}, \nabla)$ formalism.

To explain (43) we should note the general result [4] that for any generalised metric one has

$$\nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\mu\rho} = \mathrm{CoTorsion}_{\mu\nu\rho} - \mathrm{Torsion}_{\mu\nu\rho}, \tag{44}$$

where we use the metric to lower all indices. Here ω gives two covariant derivatives

$$\omega \begin{array}{ll} \theta \nearrow & \nabla \\ \\ \theta^* \searrow & {}^*\nabla \end{array}$$

depending on whether we regard θ or θ^* as the soldering form. The two are related by

$$g({}^*\nabla_X Y, Z) + g(Y, \nabla_X Z) = X(g(Y, Z)) \tag{45}$$

for vector fields X, Y, Z. The cotorsion is the torsion of ${}^*\nabla$.

Our generalisation of Riemannian geometry includes for example symplectic geometry, where the generalised metric is totally antisymmetric. So symplectic and Riemannian geometry are

included as special cases and unified in our formulation. This is what we would expect if the theory is to be the semiclassicalisation of a theory unifying quantum theory and geometry. It is also remarkable that metrics with antisymmetric part are exactly what are needed in string theory to establish T-duality, which is entirely consistent with our duality ideas of Section 2.

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